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Universal Calabi-Yau Algebra: Towards an Unification of Complex Geometry

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Abstract

We present a universal normal algebra suitable for constructing and classifying Calabi-Yau spaces in arbitrary dimensions. This algebraic approach includes natural extensions of reflexive weight vectors to higher dimensions, related to Batyrev's reflexive polyhedra, and their n -ary combinations. It also includes a 'dual' construction based on the Diophantine decomposition of invariant monomials, which provides explicit recurrence formulae for the numbers of Calabi-Yau spaces in arbitrary dimensions with Weierstrass, $K3$, etc., fibrations. Our approach also yields simple algebraic relations between chains of Calabi-Yau spaces in different dimensions, and concrete visualizations of their singularities related to Cartan-Lie algebras. This Universal Calabi-Yau Algebra is a powerful tool for decyphering the Calabi-Yau genome in all dimensions.

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1 Introduction: an Algebraic Way to Unify Calabi-Yau Geometry

Geometrical ideas play ever-increasing rôles in the quest to unify all the fundamental interactions. They were introduced by Einstein in the formulation of general relativity, and extended to higher dimensions by Kaluza and Klein in order to include electromagnetism. This is described by the geometrical principle of gauge invariance, which is also used in the formulation of the strong and weak interactions. The only known framework for combining gravity with these other interactions is String Theory, which has introduced a host of new geometrical ideas into physics.

One of the key problems in String Theory [1, 2, 3, 4] is its reduction from ten to four dimensions, which is also naturally approached from a geometrical point of view, though algebraic approaches are also possible. One of the most powerful ways of compactifying the six surplus dimensions is on a Calabi-Yau manifold, namely a complex compact manifold with Kähler structure and an $SU(3)$ holonomy group [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Extensions of the concept of Calabi-Yau manifolds to other numbers of complex dimensions are also interesting for various purposes. For example, $K3$ spaces, which may be regarded as Calabi-Yau manifolds in four real (two complex) dimensions, have been known and studied for a long time. They may be related to gauge groups using methods of algebraic geometry, via Coxeter-Dynkin diagrams [20, 21, 10, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Other useful examples are Calabi-Yau spaces in eight real (four complex) dimensions, which are used in the compactification of F theory from twelve to four dimensions.

One way of constructing at least some Calabi-Yau manifolds is as solutions of homogeneous polynomial equations in weighted projective spaces, i.e., polynomial equations in n variables whose monomial exponents $\vec{\mu}_\alpha$ satisfy the relation $\vec{k} \cdot \vec{\mu}_\alpha = d$, where the $\vec{k} = (k_1, \dots, k_{n+1})$ are $n+1$ positive weights that determine the set of monomials \vec{m}_α , and d is given by the sum of the components of k_i , i.e., $d = k_1 + \dots + k_{n+1}$. The $\vec{\mu}_\alpha = (\mu_1, \dots, \mu_{n+1})_\alpha$ are the set of $n+1$ exponents that characterize the possible monomials \mathbf{m}_α , where α runs over the set of integer monomials allowed by the Calabi-Yau equation. All Calabi-Yau spaces are not derivable from weighted projective spaces, but one can obtain many other (possibly all) Calabi-Yau manifolds as the intersections of two or more such spaces.

A useful technique for constructing Calabi-Yau spaces in any number of dimensions is to visualize the various possible monomials $\mathbf{m}_\alpha = (x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_{n+1}})_\alpha$ as the $(\mu_1, \dots, \mu_{n+1})_\alpha$ points in the Z_{n+1} integer lattice of an n -dimensional polyhedron. Using this technique, Batyrev [10] demonstrated how to associate by explicit construction a mirror polyhedron

to each Calabi-Yau space. This approach also established in a very elegant way [26] the corresponding mirror duality among Calabi-Yau spaces [31, 33].

Some topological properties of Calabi-Yau manifolds in six dimensions, namely their Euler numbers, are related to the numbers of chiral matter generations in string theory compactified on them. The Euler and Hodge numbers are directly calculable in the Batyrev approach, without solving the Calabi-Yau equations.

We have recently proposed an algebraic approach [34, 35] to the construction of Calabi-Yau manifolds, in which the geometrical structure of the Batyrev polyhedra, and therefore the associated gauge groups, are derived from an algebraic structure of the weight vectors \vec{k} . This opens the way to a generalization of Candelas' results to higher dimensions. It may also provide a deeper understanding of the origin of gauge symmetry, which appears naturally in singular limits of Calabi-Yau spaces. Moreover, our algebraic approach provides a unified framework for 'decyphering the Calabi-Yau genome', that may ultimately prove suitable for understanding which of many Calabi-Yau manifolds has been chosen by Nature to compactify string theory, if any.

In our previous articles [34, 35] we explained how to proceed constructively by first constructing the weight vectors \vec{k} in a given dimension n out of sets of between two and n vectors \vec{k} in lower dimensions. Our technique was first to extend the lower-dimensional vectors by adding zero components in any position in order to obtain n -dimensional 'extended' vectors. These were then combined in binary, ternary, etc., operations to obtain allowed n -dimensional vectors and hence Calabi-Yau spaces, which are associated into chains characterized by their eldest and youngest vectors. The eldest vectors substantially determine the structure of all vectors in the chain, and knowledge of these suffices to obtain much of the important physical information, reminiscent of the situation with eldest weight vectors and representations of Cartan-Lie algebras. The resulting Universal Calabi-Yau Algebra (UCYA) structure of reflexive weight vectors in different dimensions depends on two integer parameters: the *arity* r of the combination operation ω_r , and the dimension n .

As an example of the extension procedure in the case of $K3$ manifolds, we classified [34] the 95 different possible weight vectors \vec{k} in 22 binary chains generated by pairs of extended vectors, which included 90 of the total, and 4 ternary chains generated by triplets of extended vectors, which yielded 91 weight vectors of which 4 were not included in the binary chains. The one remaining $K3$ weight vector was found in a quaternary chain [34]. This algebraic construction provides a convenient way of generating all the $K3$ weight vectors, and arranging them in chains of related vectors whose overlaps yield further indirect relationships.

Moreover, as we discuss in more detail in this paper, our construction builds higher-

dimensional Calabi-Yau spaces systematically out of lower-dimensional ones, enabling us to enumerate explicitly their fibrations. As examples, we showed previously [34, 35] how our construction reveals elliptic and $K3$ fibrations of CY_3 manifolds. Our approach may also be used to obtain the projective weight vector structure of a mirror manifold, starting from those of a given Calabi-Yau manifold.

The main purpose of this article is to develop further the understanding of this universal algebraic approach to constructing Calabi-Yau manifolds in various numbers of dimensions with all possible structures, introducing a new technique based on the Diophantine decomposition of invariant monomials (IMs).

Our construction of a Universal Calabi-Yau algebra (UCYA) is based on the two integer parameters, *arity* and the *dimension* of the reflexive weight vectors (RWVs), that are connected one-to-one with Batyrev's reflexive polyhedra. We discussed previously how these could be classified using the natural extensions of lower-dimensional vectors and their combination via binary, ternary, etc., operations. The main innovation in this paper is the introduction of a complementary algebraic approach to the construction of Calabi-Yau spaces, based on the construction of suitable monomials $\vec{\mu}$ obeying the 'duality' condition: $\vec{k} \cdot \vec{\mu}_\alpha = d$. This new 'dual' approach is based on suitable decompositions of invariant monomials (IMs) of given dimensionality, yielding eldest vectors that could only be obtained by higher-order n -ary operations in the previous approach. This construction supplements the previous geometrical method related to Batyrev polyhedra, and enables one to calculate the numbers of eldest vectors, and hence chains, in arbitrary dimensions. We verify explicitly that the eldest vectors found in the two different ways agree in several instances for both CY_3 and CY_4 spaces, providing increased confidence in our results. The study of the Calabi-Yau equations and the associated hypersurfaces via the remarkable composite properties of IMs provides an alternative algebraic route to reflexive polyhedron techniques. We recall that the arity-dimension parameter structure is directly connected to the singularity properties of Calabi-Yau hypersurfaces, and thereby to the types of Cartan-Lie algebras. Using these remarkable properties one can hope to decypher the Calabi-Yau genome in any dimension.

We emphasize the complementarity between our approach and that of [27], which is more geometrical, being based on the classification of the all reflexive Batyrev polyhedra: see also [28]. Our method is more algebraic, being based on the construction of the weight vectors \vec{k} and/or the corresponding monomials \vec{m} . Central rôles are played in our approach by the composite structures in lower dimensions $\leq (d - 1)$ of CY d -folds, and the algebraically dual ways of expansions using weight vectors \vec{k} and invariant monomials (IMs). By analogy with the Galois normal extension of fields, we term the first way of expanding

weight vectors a *normal* extension, and the dual decomposition in terms of IMs we call the *Diophantine* expansion. These two expansion techniques are consistently combined in our algebraic approach, whose composition rules exhibit explicitly the internal structure of the Calabi-Yau algebra. Our method is closely connected to the well-known Cartan method for constructing Lie algebras, and reveal various structural relationships between the sets of Calabi-Yau spaces of different dimensions. Furthermore, this information is relatively easy to obtain without large computer facilities. We interpret our approach as revealing a ‘Universal Calabi-Yau Algebra’ [36] for the following reasons: ‘Universal’ because it may, in principle, be used to generate all Calabi-Yau manifolds of any dimension with all possible substructures, and ‘Algebra’ because it is based on a sequence of binary and higher n -ary operations on weight vectors and monomials.

We first summarize in Section 2 essential aspects of the UCYA based on normal extensions of weight vectors, as introduced in our previous articles [34, 35]. Then, in section 3 we use this method to review some results derived previously for $K3$ and CY_3 spaces, and to derive some new results for CY_4 spaces. We then develop in Section 4 our new ‘dual’ method based on the decomposition of monomials, and use it to confirm our previous results on CY_3 and CY_4 spaces. We also show that the method of invariant monomials can give recurrence formulae valid in all dimensions for Calabi-Yau spaces with specific fibrations. We illustrate this by examples of Calabi-Yau spaces of arity $r = d$, with some specific types of elliptic fibres like $\{7\}_\Delta$, $\{9\}_\Delta$, $\{10\}_\Delta$. Finally, Section 5 summarizes the conclusions and prospects of this algebraic approach.

2 The Main Elements of Universal Calabi-Yau Algebra

2.1 Basic Framework

In the search for a universal classification of Calabi-Yau spaces, one should consider the complex projective algebraic spaces CP^n with homogeneous coordinates or their quasihomogeneous generalizations, including toric varieties [37, 38], since the only complex compact submanifold (with analytic structure) embedded in C^n is a point [13]. The starting point for our algebraic approach to the classification of Calabi-Yau spaces has therefore been the construction of ‘reflexive’ weight vectors \vec{k} , whose components specify the complex quasihomogeneous projective spaces $CP^n(k_1, k_2, \dots, k_{n+1})$. These have $(n + 1)$ quasihomogeneous coordinates x_1, \dots, x_{n+1} , which are subject to the following identification:

$$(x_1, \dots, x_{n+1}) \sim (\lambda^{k_1} \cdot x_1, \dots, \lambda^{k_{n+1}} \cdot x_{n+1}). \quad (1)$$

In the case of CP^n projective spaces there exists a very powerful conjecture, called Chow's theorem, that each analytic compact (closed) submanifold in CP^n can be specified by a set of polynomial equations. The set of zeroes of quasihomogeneous polynomial equations, hereafter referred to as Calabi-Yau equations, define a projective algebraic variety in such a weighted projective space.

A d -dimensional Calabi-Yau space X_d can be given by the locus of zeroes of a transversal quasihomogeneous polynomial \wp of degree $\deg(\wp) = [d] : [d] = \sum_{j=1}^{n+1} k_j$ in a complex projective space $CP^n(\vec{k}) \equiv CP^n(k_1, \dots, k_{n+1})$ [11, 12]:

$$X \equiv X^{(n-1)}(k) \equiv \{\vec{x} = (x_1, \dots, x_{n+1}) \in CP^n(k) | \wp(\vec{x}) = 0\}. \quad (2)$$

The general quasihomogeneous polynomial of degree $[d]$ is a linear combination

$$\wp = \sum_{\vec{\mu}_\alpha} c_{\vec{\mu}_\alpha} x^{\vec{\mu}_\alpha} \quad (3)$$

of monomials $x^{\vec{\mu}_\alpha} = x_1^{\mu_{1\alpha}} x_2^{\mu_{2\alpha}} \dots x_{r+1}^{\mu_{(r+1)\alpha}}$ with the condition:

$$\vec{\mu}_\alpha \cdot \vec{k} = [d]. \quad (4)$$

This algebraic projective variety is irreducible if and only if its polynomial is irreducible. A hypersurface will be smooth for almost all choices of polynomials. To obtain Calabi-Yau d -folds one should choose reflexive weight vectors (RWVs), related to Batyrev's reflexive polyhedra or to the set of IMs. Other examples of compact complex manifolds can be obtained as the complete intersections (CICY) of such quasihomogeneous polynomial constraints:

$$X_{CICY}^{(n-r)} = \{\vec{x} = (x_1, \dots, x_{n+1}) \in CP^n | \wp_1(\vec{x}) = \dots = \wp_r(\vec{x}) = 0\}, \quad (5)$$

where each polynomial \wp_i is determined by some weight vector \vec{k}_i , $i = 1, \dots, r$.

We also recall the existence of *mirror symmetry*, relating each Calabi-Yau manifold to a dual partner, which was first observed pragmatically in the literature [7, 8, 9, 32, 33] to which we return later.

2.2 The Holomorphic-Quotient Approach to Toric Geometry

In the toric geometry approach, algebraic varieties are described by a dual pair of lattices Λ and Λ^* , each isomorphic to Z^n , and a fan Σ [37] defined on Λ_R^* , the real extension of the lattice Λ^* . In the new holomorphic-quotient approach of Batyrev [10] and Cox [38], a single homogeneous coordinate is assigned to the system specified by the variety \mathcal{U}_Σ , in a way similar

to the usual construction of CP^n . This holomorphic-quotient construction immediately gives us the usual description in terms of projective spaces, and moreover describes naturally the elliptic, $K3$ and other fibrations of Calabi-Yau spaces that interest us.

The integer points of $\Delta^* \cap \Lambda^*$ define one-dimensional cones

$$(\vec{v}_1, \dots, \vec{v}_N) = \Sigma^1_{\Delta^*}$$

of the fan Σ_{Δ^*} , to each of which one can assign a coordinate $x_k : k = 1, \dots, N$. The one-dimensional cones span the vector space Λ_R^* and satisfy $(N - n)$ linear relations with non-negative integer coefficients:

$$\sum_l k_j^l \vec{v}_l = 0, \quad k_j^l \geq 0. \quad (6)$$

These linear relations can be used to determine relations of equivalence on the space $C^N \setminus Z_{\Sigma^*}$. A variety $\mathcal{U}_{\Sigma_{\Delta^*}}$ is the space $C^N \setminus Z_{\Sigma_{\Delta^*}}$ modulo the action of a group which is product of a finite Abelian group and the torus $(C^*)^{(N-n)}$:

$$(x_1, \dots, x_N) \sim (\lambda^{k_j^1} x_1, \dots, \lambda^{k_j^N} x_N), \quad j = 1, \dots, N - n. \quad (7)$$

The set $Z_{\Sigma_{\Delta^*}}$ is defined by the fan in the following way:

$$Z_{\Sigma_{\Delta^*}} = \bigcup_I ((x_1, \dots, x_N) | x_i = 0, \forall i \in I) \quad (8)$$

where the union is taken over all index sets $I = (i_1, \dots, i_k)$ such that $(\vec{v}_{i_1}, \dots, \vec{v}_{i_k})$ do not belong to the same maximal cone in Σ^* , or several x_i can vanish simultaneously only if the corresponding one-dimensional cones \vec{v}_i are from the same cone. The elements of Σ_1^* are in one-to-one correspondence with divisors

$$D_{v_i} = \mathcal{U}_{\Sigma_{\Delta^*}^{1i}}. \quad (9)$$

The divisors D_{v_i} are subvarieties given simply by $x_i = 0$. The divisors D_{v_i} form a free Abelian group $Div(\mathcal{U}_{\Sigma_{\Delta^*}})$. In general, a divisor $D \in Div(\mathcal{U}_{\Sigma_{\Delta^*}})$ is a linear combination of some irreducible hypersurfaces with integer coefficients:

$$D = \sum a_i \cdot D_{v_i}. \quad (10)$$

The points of $\Delta \cap \Lambda$ are in one-to-one correspondence with the monomials in the homogeneous coordinates x_i . A general polynomial is given by

$$\wp = \sum_{\vec{\mu} \in \Delta \cap \Lambda} c_{\vec{\mu}} \prod_{l=1}^{\Lambda^*} x_l^{\langle \vec{v}_l, \vec{\mu} \rangle + 1}. \quad (11)$$

The equation $\wp = 0$ is well defined, and \wp is holomorphic if the following condition

$$\langle \vec{v}_l, \vec{\mu} \rangle \geq -1 \text{ for all } l \quad (12)$$

is fulfilled. The $c_{\vec{\mu}}$ parametrize a family of M_{Δ} of Calabi-Yau surfaces defined by the zero locus of \wp .

We see that toric varieties can be defined by the quotient $C^k \backslash Z_{\Sigma}$, not only by a group $(C \backslash 0)^{k-n}$. One should divide $C^k \backslash Z_{\Sigma}$ also by a finite Abelian group $G(v_1, \dots, v_k)$, that is determined from the relations between the D_{v_i} divisors. In this case, the toric varieties can often have orbifold singularities, $C^k \backslash G$. Orbifolding of a manifold M by a group G gives a new orbit space

$$M \Rightarrow M/G \quad (13)$$

which is characterized by the identification $x \sim y$ of all points $x, y \in M$, such that

$$x = g(y), \quad g \in G. \quad (14)$$

The isotropy subgroup G_P is determined by the following condition:

$$G_{x_P} = \{g \in G : g \cdot x_P = x_P, \quad x_P \in M\} \in G. \quad (15)$$

These points are called fixed points, and yield singularities of the hypersurface M/G . If G acts freely, i.e., if $g \cdot x_P = x_P$ for some $g \in G$ implies $g = 1$, then the projection

$$p : M \Rightarrow M/G \quad (16)$$

is called a covering projection.

Similarly, due to the transformations $x_i \rightarrow \lambda^{k_i} x_i$, $\lambda \in C^*$, whose orbits define points of CP^n , the weighted projective space has singular strata

$$F_I = P^n(\vec{k}) \cap [x_i = 0, \forall i \in (1, \dots, n+1) \setminus I]$$

if the subset of $([k_i], i \in I)$, has a non-trivial common factor N_I . The possible singular sets on X are either points or curves. In the cases of singular points, these singularities are

locally of the type C^2/G , whilst a singular curve has locally a C^3/G singularity, where G is a discrete group. Both types of singularities and their resolution can be described by the methods of toric geometry, using the ‘blow-up’ procedure [11, 12, 30]. For example, resolving the C^2/Z_n singularity gives for rational, i.e., genus zero, (-2) -curves an intersection matrix that coincides with the A_{n-1} Cartan matrix. For a general form of the C^2/G singularity, one can show [11] using the condition $K = 0$ that G is a discrete subgroup of $SU(2)$. Any discrete subgroup of $SU(2)$ can be projected into a subgroup of $SO(3)$, and thus can be related to the finite symmetry classification of three-dimensional space. Thus, resolving the orbifold singularities yields a beautiful interrelation between the classification of finite group rotations in three-space and the ADE classification of Cartan-Lie algebras.

2.3 Calabi-Yau Spaces as Toric Fibrations

A huge set of the reflexive polyhedra corresponding to Calabi-Yau manifolds can be classified by their fibration structures. In this way it is possible, as we show explicitly later, to connect the structures of all the projective vectors in one specific dimension with the projective vectors of other dimensions and, as a result, to construct a new algebra acting on the set of all reflexive weight vectors (RWVs), giving the full set of CY_d hypersurfaces in all dimensions: $1 \leq d < \infty$.

To understand this better, we provide more information about two operations, *intersection* and *projection*, that give information about the possible fibration structures of CY_d spaces defined via reflexive polyhedra [26, 29]:

- There may exist a projection operation $\pi : \Lambda \rightarrow \Lambda_{n-k}$, where Λ_{n-k} is an $(n - k)$ -dimensional sublattice and $\pi(\Delta)$ is also a reflexive polyhedron;
- There may exist an intersection operation σ through the origin of a reflexive polyhedron, such that $\sigma(\Delta)$ is again a $(n - l)$ -dimensional reflexive polyhedron;
- These operations exhibit the following duality properties:

$$\begin{aligned} \pi(\Delta) &\Leftrightarrow \sigma(\Delta^*) \\ \sigma(\Delta) &\Leftrightarrow \pi(\Delta^*). \end{aligned} \tag{17}$$

For a reflexive polyhedron Δ with fan Σ^* over a triangulation of the facets of Δ^* , the Calabi-Yau hypersurface in the variety \mathcal{U}_{Σ^*} is given by the zero locus of the polynomial \wp . One can consider the variety \mathcal{U}_{Σ^*} as a fibration over the base $\mathcal{U}_{\Sigma_{base}}$ with generic fibre $\mathcal{U}_{\Sigma_{fibre}}$.

This fibration structure can be written in terms of homogeneous coordinates. The fibre is determined as an algebraic subvariety by the polyhedron $\Delta_{fibre}^* \subset \Delta_{CY}^*$, and the base can be seen as projection of the fibration along the fibre. The set of one-dimensional cones in Σ_{base} (the primitive generator of a cone is zero or \tilde{v}_i) is the set of images of one-dimensional cones in Σ_{CY} (the primitive generator is \vec{v}_j) that do not lie in N_{fibre} . The image Σ_{base} of Σ_{CY} under

$$\Pi : N_{CY} \rightarrow N_{base}$$

gives us the following relation:

$$\Pi \vec{v}_i = r_i^j \cdot \tilde{v}_j \quad (18)$$

if $\Pi \vec{v}_i$ is in the set of one-dimensional cones determined by \tilde{v}_j $r_i^j \in N$, otherwise $r_i^j = 0$.

Similarly, the base space is the weighted space with the toroidal structure:

$$(\tilde{x}_1, \dots, \tilde{x}_{\tilde{N}}) \sim (\lambda^{\tilde{k}_j^1} \cdot \tilde{x}_1, \dots, \lambda^{\tilde{k}_j^{\tilde{N}}} \cdot \tilde{x}_{\tilde{N}}), \quad j = 1, \dots, \tilde{N} - \tilde{n}, \quad (19)$$

where the \tilde{k}_j^i are integers such that $\sum_j \tilde{k}_i^j \tilde{v}_j = 0$. The projection map from the variety \mathcal{U}_Σ to the base can be written as

$$\tilde{x}_i = \prod_j x_j^{r_j^i}, \quad (20)$$

that corresponds to a redefinition of the torus transformation for \tilde{x}_i :

$$\Pi : \tilde{x}_i \rightarrow \lambda^{k_l^j \cdot r_j^i} \cdot \tilde{x}_i, \quad \sum k_l^j \cdot r_j^i \cdot \tilde{v}_i = 0. \quad (21)$$

A simple well-known example with an elliptic fibre and base P^1 is given by the following Weierstrass equation for the fibre. To illustrate how this may appear, we consider the case of a dual pair of polyhedra $\Delta(P^3(1, 1, 4, 6)[12])$ and its dual Δ^* . The mirror polyhedron contains, as intersection H through the interior point, the elliptic fibre $P^2(1, 2, 3)$. For all integer points of Δ^* (excluding the interior point) one can write the corresponding complex variables:

$$\begin{aligned} \vec{v}_1 &= (0, \underline{-2, -3}) \rightarrow x_1 \\ \vec{v}_2 &= (0, \underline{-1, -2}) \rightarrow x_2 \\ \vec{v}_3 &= (0, \underline{-1, -1}) \rightarrow x_3 \end{aligned}$$

$$\begin{aligned}
\vec{v}_4 &= (0, \underline{0}, -1) \rightarrow x_4 \\
\vec{v}_0 &= (0, \underline{0}, 0) \\
\vec{v}_6 &= (0, \underline{1}, 0) \rightarrow x_6 \\
\vec{v}_7 &= (0, \underline{0}, 1) \rightarrow x_7
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\vec{v}_8 &= (-1, \underline{-4}, -6) \rightarrow x_8 \\
\vec{v}_9 &= (1, \underline{0}, 0) \rightarrow x_9.
\end{aligned} \tag{23}$$

There are some linear relations between integer points inside the fibre:

$$\begin{aligned}
\vec{v}_1 + 2 \cdot \vec{v}_6 + 3 \cdot \vec{v}_7 &= 0, \\
\vec{v}_2 + \vec{v}_6 + 2 \cdot \vec{v}_7 &= 0, \\
\vec{v}_3 + \vec{v}_6 + \vec{v}_7 &= 0, \\
\vec{v}_4 + \vec{v}_7 &= 0
\end{aligned} \tag{24}$$

and we have the relation between all the points in Δ^* :

$$\vec{v}_8 + \vec{v}_9 + 4 \cdot \vec{v}_6 + 6 \cdot \vec{v}_7 = 0. \tag{25}$$

We see later that, according our algebraic approach, this example corresponds to the case of the eldest RWV $\vec{k}_8 = (1, 1, 1, 1, 1, 1, 8, 13)$ [27], which defines a CY_6 space that has an elliptic K3 fibre. The 7-dimensional polyhedron corresponding to this CY_6 has an intersection consisting of the 3-dimensional polyhedron $\Delta(P^3(1, 1, 4, 6))$. This example indicates that the toric description ‘closes upon itself’ and has a natural extension to higher dimensions. This provides additional motivation for an algebraic description of Calabi-Yau geometry in all dimensions. The toric holomorphic approach can be embedded naturally in the UCYA: see [34, 35] and Sections 3 and 4.

The polyhedron $\Delta(P^3(1, 1, 4, 6))$ contains 39 points, which can be subdivided as follows. The even points of the fibre $P^2(1, 2, 3)$ are determined by the intersection of the plane $m_1 + 2 \cdot m_2 + 3 \cdot m_3 = 0$ with the positive integer lattice. This plane separates the remaining 32 points into 16 ‘left’ and 16 ‘right’ points. These ‘left’ and ‘right’ points each have structures isomorphic to the Coxeter-Dynkin diagrams for E_8 , and their singularities can be used to produce these groups, as we now illustrate.

The 7 points of the plane $H(\Delta) = m_1 + 2m_2 + 3m_3$ are the following:

$$\begin{aligned}
t_1 &= (5, \underline{-1, -1}) \rightarrow (x_8^6 x_9^6) \cdot (\underline{x_1^6 x_2^4 x_3^3 x_4^2}), \\
t_2 &= (3, \underline{0, -1}) \rightarrow (x_8^4 x_9^4) \cdot (\underline{x_1^4 x_2^3 x_3^2 x_4^2 x_6}), \\
t_3 &= (2, \underline{-1, 0}) \rightarrow (x_8^3 x_9^3) \cdot (\underline{x_1^3 x_2^2 x_3^2 x_4 x_7}), \\
t_4 &= (1, \underline{1, -1}) \rightarrow (x_8^2 x_9^2) \cdot (\underline{x_1^2 x_2^2 x_3 x_4^2 x_6^2}), \\
t_5 &= (0, \underline{0, 0}) \rightarrow (x_8 x_9) \cdot (\underline{x_1 x_2 x_3 x_4 x_6 x_7}), \\
t_6 &= (-1, \underline{2, -1}) \rightarrow (\underline{x_2 x_4^2 x_6^3}), \\
t_7 &= (-1, \underline{-1, 1}) \rightarrow (\underline{x_3 x_7^2}).
\end{aligned} \tag{26}$$

One can write the Weierstrass equation for the E_{8_L} group based on the polyhedron $\Delta(P^3(1, 1, 4, 6))$ in the form:

$$\begin{aligned}
&\underline{x}_6^3 + \underline{x}_6^2 \cdot (a_2^{(1)} x_8 x_9^3 + a_2^{(2)} x_9^4) + \\
&\underline{x}_1^4 \cdot \underline{x}_6 \cdot (a_4^{(1)} x_8^3 x_9^5 + a_4^{(2)} x_8^2 x_9^6 + a_4^{(3)} x_8 x_9^7 + a_4^{(4)} x_9^8) + \\
&\underline{x}_1^6 \cdot (a_6^{(1)} x_8^5 x_9^7 + a_6^{(2)} x_8^4 x_9^8 + a_6^{(3)} x_8^3 x_9^9 + a_6^{(4)} x_8^2 x_9^{10} + a_6^{(5)} x_8 x_9^{11} + a_6^{(6)} x_9^{12}) \\
&= \underline{x}_7^2 + a_1 \cdot \underline{x}_6 \underline{x}_7 \cdot x_9^2 + \underline{x}_7 \cdot (a_3^{(1)} x_8^2 x_9^4 + a_3^{(2)} x_8 x_9^5).
\end{aligned} \tag{27}$$

The second Weierstrass equation for the E_{8_R} group can be obtained from this equation by interchanging base variables: $x_8 \leftrightarrow x_9$. The coefficients a_i correspond to the notations of the paper by Bershadsky *et al.* [22]. The Weierstrass triangle equation can be presented in the following general form, where we write $\underline{x}_6 \equiv x$, $\underline{x}_7 \equiv y$:

$$y^2 + a_1 \cdot x \cdot y + a_3 \cdot y = x^3 + a_2 \cdot x^2 + a_4 \cdot x + a_6, \tag{28}$$

and the a_i are polynomial functions on the base. In a more simplified form, the Weierstrass equation can be written as:

$$y^2 = x^3 + x \cdot f + g, \tag{29}$$

with discriminant

$$\Delta = 4f^3 + 27g^2. \tag{30}$$

In the zero locus of the discriminant, some divisors D_i define the degeneration of the torus-fibre. If one can choose the polynomials f and g to be homogeneous of orders 8 and 12,

respectively, the fibration will degenerate over 24 points of the base. For this form of Weierstrass equation, there exists an *ADE* classification of degenerations of elliptic fibres, as given by Kodaira [22]. In this approach, the type of degeneration of the fibre is determined by the orders a, b, c of the zeroes of the functions f, g and δ . For the case under consideration, one has two singularities at $x_8 = 0$ and $x_9 = 0$ with $(a = 4, b = 5, c = 10)$, both corresponding to E_8 . The general algorithm for the *ADE* classification of elliptic singularities for the general Weierstrass equation was considered by Tate [23, 22]. Tate's algorithm allows one to define in general the type of Lie-algebra singularity. The example of the $K3$ space determined by the RWV $\vec{k}_4 = (1, 1, 3, 4)$ [9] is displayed in Fig. 1.

As an example of the use of these observations, we consider F -theory, which can be considered as a decompactification of the type-*IIA* string. Our understanding of the duality between the heterotic string and type-*IIA* string in $D = 6$ dimensions can be used in the duality between heterotic string on T^2 and F -theory on an elliptically fibred $K3$ hypersurface. The gauge group is directly defined by the above *ADE* classification of the quotient singularities of hypersurfaces. The Cartan matrix of the Lie group in this case coincides, up to a sign, with the intersection matrix of the blown-down divisors. There are two different mechanisms for enhancing the gauge groups on the F -theory side and on the heterotic string theory side. On the F -theory side, the singularities of the Calabi-Yau hypersurface give rise to the gauge groups, but on the heterotic side the singularities can enhance the gauge group if 'small' instantons of the gauge bundle lie on these singularities. These questions have been studied as a function of the number of instantons placed on a singularity of type G , where G is a simply laced group.

Furthermore, the elliptic CY_n ($n = 3, 4$) with $K3$ fibres can be also used to study F -theory dual compactifications of the $E_8 \times E_8$ or $SO(32)$ - string theory. To study this in toric geometry, one may consider a $K3$ -polyhedron fibre as a subpolyhedron of the CY_n polyhedron, and the Dynkin diagrams of the gauge groups of the type-*IIA* string (F -theory) compactifications on the corresponding threefold (fourfold) can be seen in the polyhedron of this $K3$ hypersurface. Of course, one could also consider the case of an elliptic CY_4 -fold with CY_3 fibre, where the latter is a Calabi-Yay hypersurface with $K3$ fibre.

We recall [34] that the lattice structure of the $K3$ projective vectors obtained by a binary construction exhibits a very interesting correspondence between the Dynkin diagrams for Cartan-Lie groups in the A, D series and $E_{6,7,8}$ and particular reflexive weight vectors (see also Figure 1):

$$\begin{aligned}
\vec{k}_1 = (1) &\leftrightarrow A_r; \\
\vec{k}_2 = (1, 1) &\leftrightarrow D_r; \\
\vec{k}_3 = (1, 1, 1) &\leftrightarrow E_6; \\
\vec{k}_3 = (1, 1, 2) &\leftrightarrow E_7; \\
\vec{k}_3 = (1, 2, 3) &\leftrightarrow E_8.
\end{aligned} \tag{31}$$

This appearance in Calabi-Yau geometry of the A, D and E series of Cartan-Lie algebras is connected [34] with specific one- (two-) (three-)dimensional structures in an auxiliary complex space.

2.4 The Arity-Dimension Structure of Universal Calabi-Yau Algebra

Our objective is to construct an universal algebra [36] acting on the set of reflexive weight vectors in all dimensions, $A_n \equiv \{\text{RWV}(n)\}$, and the corresponding set of invariant monomials, $\{\text{IMs}(n)\}$, which is ‘dual’ to A_n in the sense of (4). We note that the number of IMs is much less the full set of monomials $\vec{m}_\alpha : 1 \leq \alpha \leq \alpha_{max}$ which determine the Calabi-Yau equation. Through the IMs one can determine the highest vectors of the chains and also the full list of weight vectors in the corresponding chain. To see this, we start from the unit IM in some dimension n and then, via a Diophantine expansion, can go on to determine the conic IMs, the cubic IMs, the quartic IMs, etc.. Similarly, one can continue this process of studying the set of IMs via the Diophantine expansions of conic IMs, of cubic IMs, etc..

The RWVs and IMs provide independent routes for constructing explicitly Calabi-Yau spaces of arbitrary dimension (including CICYs). The resulting UCYA structure of RWVs in different dimensions depends on two integer parameters, including the ‘arity’ r defined below, as well as the dimension n . An overview in the (n, r) plane is shown in Fig. 2, where the entries $A_n^{(r)}$ label the types of possible eldest vectors, corresponding to ‘chains’ of related Calabi-Yau spaces.

As we have just discussed, the algebraic-geometry realization [11, 12, 26] of Coxeter-Dynkin diagrams provides a general characterization of the possible structures in singular limits of Calabi-Yau hypersurfaces. Thus, a deeper understanding of the origins of gauge invariance provides an additional motivation for studying string vacua via our unification of the complex geometry of $d = 1$ elliptic curves, complex tori, $K3$ manifolds, CY_3 , CY_4 , etc. This point is illustrated in Fig. 2, where the points on the the first three sloping lines,

labelled A_r (red), D_r (green) and E (blue), correspond to those d -folds that are characterized by the ‘maximal’ quotient A, D, E singularities, respectively ¹. As we discuss later in more detail, this characterization of the types of singularities is directly connected to the degrees of the associated monomials - linear, conics, cubics, quartics, etc., that appear along the corresponding sloping lines.

We now consider in detail important operations in the (r, n) plane. We seek a map from the sets, A_i , of reflexive weight vectors (RWVs) of lower dimensions $i = 1, 2, 3, \dots, p < n$ to the set of RWVs of dimension n , A_n . For this purpose, we define on the space $A_p = \cup_{i=1}^p A_i$ of RWVs a set $\omega_r : (p = n - r + 1)$ of operations combining r elements, where $r = 0, 1, 2, 3, \dots, n$:

$$\omega_r : \underbrace{A_p \star A_p \star \dots \star A_p}_r \mapsto A_n. \quad (32)$$

We term r the *arity* (or rank) of ω_r [36]. An operation of arity $r = 0$ on A , called a *nullary* (or constant) operation, selects one element of A . An operation of arity $r = 1$ on A is just a mapping of A into A , termed a *unary* operation, and higher-order operations with arity $r = 2(3)$ are binary (ternary) operations, etc.. The precise sense of the symbol “ \star ” will be provided later, together with a description of the normal property of the algebra. It is possible to generate all n -dimensional weight vectors if one already knows all the RWVs of lower dimensions $1, 2, \dots, n - 1$, using the unary, binary, ternary, ... , $r = n$ -ary composition operations ω_r .

The first step in this programme is to define a sequence of unary operations consisting of extensions of the RWVs $\vec{k}_i \in A_i : i < n$ to n -dimensional RWVs $\in A_n$: $A_i \mapsto \{A_i^{ex}\}_{(n)}$, obtained simply by adding one or more zero components to this vector in any location (see Figure 2):

$$\begin{aligned} \{A_{n-1} : \vec{k}_{(n-1)}\} &= (\dots\dots\dots) \rightarrow \{\{A_{n-1}^{ex}\}_n : \vec{k}_n^{(ex)} = (\dots, 0, \dots) + \text{all permutations}\} \\ \{A_{n-2} : \vec{k}_{(n-2)}\} &= (\dots\dots\dots) \rightarrow \{\{A_{n-2}^{ex}\}_n : \vec{k}_n^{(ex)} = (\dots, 0, \dots, 0, \dots) + \text{all permutations}\} \\ \{A_{n-3} : \vec{k}_{(n-3)}\} &= (\dots\dots\dots) \rightarrow \{\{A_{n-3}^{ex}\}_n : \vec{k}_n^{(ex)} = (\dots, 0, \dots, 0, \dots, 0, \dots) + \text{all permutations}\} \\ \dots\dots\dots &= \dots\dots\dots \end{aligned} \quad (33)$$

In this first step, the extensions of all reflexive vectors of orders $1, 2, \dots, i, \dots, (n-1)$ define sets of extended weight-vectors \mathcal{A}_n^i corresponding to operations ω_r of arity $r = n+1-i = 2, 3, \dots, n$:

¹To be more precise, the D line includes also A -type singularities, and the E line includes also D -type and A -type singularities.

$$\begin{aligned}
(A_1, A_2, A_3, \dots, A_{n-1}) &\mapsto \cup_{i=1}^{i=n-1} \{A_i^{ex}\}_{(n)} \equiv \mathbb{A}_{(n)}^{(n-1)}, \\
(A_1, A_2, A_3, \dots, A_{n-2}) &\mapsto \cup_{i=1}^{i=n-2} \{A_i^{ex}\}_{(n)} \equiv \mathbb{A}_{(n)}^{(n-2)}, \\
&\dots\dots\dots = \dots\dots\dots \\
(A_1, A_2, A_3) &\mapsto \cup_{i=1}^{i=3} \{A_i^{ex}\}_{(n)} \equiv \mathbb{A}_{(n)}^{(3)} \\
(A_1, A_2) &\mapsto \cup_{i=1}^{i=2} \{A_i^{ex}\}_{(n)} \equiv \mathbb{A}_{(n)}^{(2)} \\
(A_1) &\mapsto \{A_1^{ex}\}_{(n)} \equiv \mathbb{A}_{(n)}^{(1)},
\end{aligned} \tag{34}$$

with the following embeddings

$$\mathbb{A}_{(n)}^{(1)} \subset \mathbb{A}_{(n)}^{(2)} \subset \mathbb{A}_{(n)}^{(3)} \subset \dots \subset \mathbb{A}_{(n)}^{(n-2)} \subset \mathbb{A}_{(n)}^{(n-1)} \tag{35}$$

defined naturally.

The second step consists of a set of m -ary operations (where the arity $m = 2, 3, \dots, r_{max} = n$) to get the complete list of the RWVs of dimension n :

$$\begin{aligned}
arity = 2 : \quad & \left(\mathbb{A}_{(n)}^{(n-1)} \right)^2 = \mathbb{A}_{(n)}^{(n-1)} \star \mathbb{A}_{(n)}^{(n-1)} \implies (A_n^2) \\
arity = 3 : \quad & \left(\mathbb{A}_{(n)}^{(n-2)} \right)^3 = \mathbb{A}_{(n)}^{(n-2)} \star \mathbb{A}_{(n)}^{(n-2)} \star \mathbb{A}_{(n)}^{(n-2)} \implies (A_n^3) \\
arity = & \dots\dots\dots \\
arity = (n-2) : & \left(\mathbb{A}_{(n)}^{(3)} \right)^{(n-2)} = \mathbb{A}_{(n)}^{(3)} \star \dots \star \mathbb{A}_{(n)}^{(3)} \implies (A_n^{n-2}) \\
arity = (n-1) : & \left(\mathbb{A}_{(n)}^{(2)} \right)^{(n-1)} = \mathbb{A}_{(n)}^{(2)} \star \dots \star \mathbb{A}_{(n)}^{(2)} \implies (A_n^{n-1}) \\
arity = n : & \left(\mathbb{A}_{(n)}^{(1)} \right)^n = \mathbb{A}_{(n)}^{(1)} \star \dots \star \mathbb{A}_{(n)}^{(1)} \implies (A_n^n)
\end{aligned} \tag{36}$$

where A_n^r is the set of RWVs with arity r , and $A_n^1 \equiv A_n$.

The specification of the algebraic formalism for the expansion technique is then completed by defining the symbol \star appearing in the above r -ary composition operations on the sets $\mathbb{A}_{(n)}^{(n-r+1)}$:

$$\omega_r : \underbrace{\mathbb{A}_n^{(n-r+1)} \star \mathbb{A}_n^{(n-r+1)} \star \dots \star \mathbb{A}_n^{(n-r+1)}}_r \implies A_n^r. \quad (37)$$

The symbol \star defines the ‘intersection’ of r different extended weight vectors, in the sense that they share a common set of invariant monomials $\vec{\mu}_\alpha$ satisfying simultaneously the r conditions:

$$\vec{\mu}_\alpha \cdot \vec{k}_n^{i(ex)} = d_i, \quad (38)$$

where $i = 1, 2, \dots, r$.

We term the ω_r operation on the set $\mathbb{A}_n^{(n-r+1)}$ *normal* if the intersections $\vec{k}_n^{(ex)} \in \mathbb{A}_{(n)}^{(n-r+1)}$ of some r extended weight vectors, where $2 \leq r \leq n$, give a *normal object* in the sense that:

$$\begin{array}{llll} \text{if} & \bigcap_{i=1}^{i=r} \left(\vec{k}_n^{(ex)} \right)^{(i)} & = & \{ \square^{ref} \}_{n-r} \quad \left(\vec{k}_n^{(ex)} \right)^{(i)} \in \mathbb{A}_{(n)}^{(n-r+1)}, \\ \text{then} & \bigcup_{i=1}^{i=r} \left(\vec{k}_n^{(ex)} \right)^{(i)} & = & \vec{k}_n, \quad \{ \vec{k}_n \}_R \in A_n^r. \end{array} \quad (39)$$

The term *normal* is appropriate for the following reasons. One is that the extensions of all the RWVs in lower dimensions $1 \leq i \leq (n-1)$ to the next dimension n give the complete list of RWVs of dimension n , just as the usual normal Galois extension of a field K includes all the roots of polynomial equations: if a polynomial over the field K has one root in the normal extension P , all the roots must be in the field P . Another reason why the term *normal* is appropriate is connected with our conjecture that this extension is also complete under mirror duality, i.e., all the mirror CY_d spaces with $d = n-2$ can also be expressed in terms of RWVs with lower dimensions $1 \leq i \leq n$ or their extended vectors. This conjecture will be illustrated later in the case of $K3$ mirror partners.

We use below two examples of such a *normal object* $\{ \square^{ref} \}_{n-r}$, one of which can be identified geometrically as a reflexive Batyrev polyhedron, and the other algebraically as a set of invariant monomials (IMs). An analysis in which the symbol $\{ \square^{ref} \}_{n-r}$ is identified as a Batyrev polyhedron [34] is illustrated in the Fig. 3. It served as our first way of identifying the 4242 chains of CY_3 spaces with arity 2, and we use it here to derive some new results for CY_4 spaces. A second way of deriving these results, where the symbol $\{ \square^{ref} \}_{n-r}$ is identified with sets of IMs, is discussed in Section 3. There we use the IM approach to rederive previous results for $K3$ and CY_3 spaces, to obtain further results for CY_4 , and to derive some results for arbitrary dimension n .

In the IM case, the symbol $\cup..$ means a simple linear algebraic sum of the extended RWVs, $\vec{k}_n^{(ex)} \in \{\mathcal{A}\}_n^{(n-r+1)}$, which give the following set:

$$\bigcup_{i=1}^{i=r} \left(\vec{k}_n^{(ex)} \right)^{(i)} = m_1 \left(\vec{k}_n^{(ex)} \right)^{(1)} + \dots + m_r \left(\vec{k}_n^{(ex)} \right)^{(r)} = \left\{ \vec{k}_n \right\}_R \subset A_n^r, \quad (40)$$

where $m_1 \geq 1, \dots, m_r \geq 1$ are positive integers that determine a whole chain, $\{\vec{k}_n\}_R$, of RWVs with some common properties. In general this normal ω_r operation gives the structure of the chain and directly determines the eldest vector, corresponding to the minimal values, $m_1 = m_2 = \dots = m_r = 1$. To determine the complete listing of the chain, we must specify all these coefficients m_i . As discussed later, this may be done using mirror duality or a Diophantine expansion.

An operation ω_r that yields such a normal object is a special r -ary operation that yields specific algebraic relations between the RWVs in different dimensions, defining a ‘dual’ relation between the \cup and \cap operations. In order to specify correctly such an ω_r operation, one should study more carefully the sets $\{\mathcal{A}\}_n^{(n-r+1)}$.

First, one should use in ω_r only r independent extended RWVs. Independence here means that one should take only those r extended vectors whose intersections give an object of dimension $n - r$. This may conveniently be done by constructing from the r extended RWVs the $r \times n$ rectangular matrix $A_{r \times n}$ and to check that the determinant of the $r \times r$ matrix $A_{r \times n} \cdot A_{r \times n}^T$ does not vanish.

Secondly, there are requirements concerning the self-consistency of the algebra with respect to the algebraic operations of different arities. This is familiar in mathematics from the theories of rings and fields, in which two binary operations exist, and there are additional conditions on the simultaneous actions of these two operations, such as the law of distributivity. We term the following analogous property of this algebra, which plays an important role for arities $r > 2$ also as ‘distributivity’: If a RWV can be obtained by two or more operations with different arities, e.g., with arities 2 and 3, this vector will be considered a ‘good’ weight vector only if it is determined correctly by all arity operations.

We must impose this requirement because even weight vectors which can be got from ‘good’ arity 3, 4, \dots , r operations may actually not be reflexive, because they cannot be obtained legitimately by constructions of lower arity 2, 3, \dots , $r - 1$. We call them *false* vectors. So, if a weight vector is ‘good’ from the point of view of arity 2, it cannot be ‘false’ from the point of view of the higher arities, $\omega_r \geq 3$. Also, if a weight vector is ‘good’ from the viewpoint of arity 3, and has no arity 2 structure, it cannot be ‘false’ with respect to the

arities $\omega_r \geq 4$ etc. In other words, there is an ‘ordering’ of the arity operations, according to which lower arities ‘control’ higher arities. We exhibit later some examples of ‘false’ vectors in the CY_3 case.

Thirdly, we must clarify the notions of reducible and irreducible chains and eldest vectors. Sometimes, one finds by operations with the same arity r two or more chains/eldest vectors with the same intersection structure and, moreover, one chain can be expressed in terms of another chain, i.e.:

$$ak_1^{ex} + bk_2^{ex} + ck_3^{ex} = \tilde{a}\tilde{k}_1^{ex} + \tilde{b}\tilde{k}_2^{ex} + \tilde{c}\tilde{k}_3^{ex}, \quad (41)$$

where all the coefficients a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$ are positive integers. There exists a ‘minimal’ basis in terms of which all reducible chains can be constructed as linear combinations with positive integer coefficients. This issue is transparent in the IM method, which is powerful enough to provide more solutions than we need to determine the chains. For example, there is an irreducible chain of CY_4 spaces of arity $\omega_r = 4$, corresponding to the third E_r line, that can be constructed from the following extended vectors,

$$\begin{aligned} \vec{k}^{1(ex)} &= (0, 0, 1, 0, 1, 0) = e_1, \\ \vec{k}^{2(ex)} &= (1, 0, 0, 2, 3, 0) = e_2, \\ \vec{k}^{3(ex)} &= (0, 1, 0, 2, 3, 0) = e_3, \\ \vec{k}^{4(ex)} &= (0, 0, 1, 0, 0, 1) = e_4, \end{aligned} \quad (42)$$

Other sets of weight vectors generate the same chain, one example being

$$\begin{aligned} \vec{k}'^{1(ex)} &= (1, 0, 0, 2, 3, 0) = e'_1 = e_2, \\ \vec{k}'^{2(ex)} &= (0, 1, 0, 2, 3, 0) = e'_2 = e_3, \\ \vec{k}'^{3(ex)} &= (0, 0, 3, 0, 1, 2) = e'_3 = e_1 + 2e_4, \\ \vec{k}'^{4(ex)} &= (0, 0, 3, 0, 2, 1) = e'_4 = 2e_1 + e_4, \end{aligned} \quad (43)$$

One can easily check that the intersections are the same for these two chains, and that $\sum m'_i \vec{k}'^{i(ex)} \subset \sum m_j \vec{k}^{j(ex)}$, where $i, j = 1, \dots, 4$.

The choice $m_1 = 1, m_2 = 1$ is always possible for the binary case, and determines what we term the *eldest* RWV in the chain. With the above clarification, in general there is also a unique eldest RWV for good operations with arity $r > 2$. There also exists a set of coefficients m_i which determines the *youngest* RWV [34] in the chain. The set of possible values for the coefficients m_i determining a chain is easiest to find for the double chains with arity $r = 2$. The maximal values for m_1 and m_2 are determined by the dimensions of the extended vectors $\left(\vec{k}_n^{ex}\right)^{(2)}$ and $\left(\vec{k}_n^{ex}\right)^{(1)}$, respectively. There are analogous rules for triple and higher-order chains [34]. The RWVs inside a chain can also be enumerated using the technique of decomposing the IM that determines the chain [35].

Thus, in the framework of the UCYA, it is enough to know the arity-dimension structure for the eldest RWVs, as seen in Fig. 3, just as the eldest weights determine representations of Cartan-Lie algebras. The tremendous numbers of RWVs and/or reflexive polyhedra may then be found in any specific cases of interest.

We have seen above how the composition operations ω_r with $2 \leq r \leq n$ give a universal, normal and self-consistent map of the set of RWVs of lower dimensions $1, 2, \dots, (n - r + 1)$ into to the set of n -dimensional RWVs:

$$\omega_r : \quad \vec{k}_1, \vec{k}_2, \dots, \vec{k}_{(n-r+1)} \quad \Longrightarrow \quad \vec{k}_n, \quad 2 \leq r \leq n. \quad (44)$$

Thus, a given $RWV(n) : \vec{k}_n \in A_n$ can be obtained in $\leq (n - 1)$ different ways, depending on the arity structure. More exactly, constructions with arity $r = 2$ correspond to the structure of $(n - 1)$ -dimensional sections of the set of RWVs or reflexive polyhedra, and constructions with arity $r = n$ correspond to the trivial point sections of the RWVs or reflexive polyhedra. The union of all such subsets should give the full list of all n -dimensional RWVs with all possible internal structures, as seen in Fig. 2:

$$\bigcup_{r=2}^{r=n} (A_n^r) = (A_n^{(1)}) \equiv (A_n). \quad (45)$$

The extension mechanism provides the ω_r composition operations of this algebra for all possible arities $2 \leq r \leq n$. This extension is normal, because we consider d -dimensional objects which have definite universal structure properties common for all dimensions. The condition for such a normal extension applies if there are composition operations that allow one to reconstruct objects with every possible internal structure in any dimension d , if one has already knows their structures in lower dimensions $(d - 1), (d - 2), \dots, 1$. Indeed, information in only the lowest dimension is sufficient. Conversely, it is possible to determine algebraically the properties of objects in lower dimensions, starting from objects in higher dimensions.

The restriction that the geometrical structure of such objects in any given dimension d remembers the structures of objects in lower dimensions, $(d-1), (d-2), \dots, 1$ is closely connected with the intersection-projection mirror duality: $\sigma \leftrightarrow \pi$. Concretely, it means that for any $p : d > p > 0$, there should exist at least one $(d-p)$ intersection or $(d-p)$ projection in which one can see a $(d-p)$ -dimensional object with similar properties. This property is reflected in the following ‘normal’ expansion series and associated ‘factor’ set:

$$\begin{aligned}
& \mathcal{U}_0 \triangleleft \mathcal{U}_1 \triangleleft \mathcal{U}_2 \triangleleft \dots \mathcal{U}_n \triangleleft \dots & (46) \\
& \hat{\sigma} \{ \mathcal{U}_1 \} = \mathcal{U}_0 & or & \hat{\pi} \{ \mathcal{U}_1 \} = \mathcal{U}_0 \\
& \hat{\sigma} \{ \mathcal{U}_2 \} = \mathcal{U}_1 & or & \hat{\pi} \{ \mathcal{U}_2 \} = \mathcal{U}_1 \\
& \hat{\sigma} \{ \mathcal{U}_3 \} = \mathcal{U}_2 & or & \hat{\pi} \{ \mathcal{U}_3 \} = \mathcal{U}_2 \\
& & \dots & \dots \\
& \hat{\sigma} \{ \mathcal{U}_d \} = \mathcal{U}_{d-1} & or & \hat{\pi} \{ \mathcal{U}_d \} = \mathcal{U}_{d-1} \\
& & \dots & \dots
\end{aligned} \tag{47}$$

where we denote by $\hat{\sigma} \{ \dots \}$ the $(d-1)$ -dimensional intersection and by $\hat{\pi} \{ \dots \}$ the $(d-1)$ -dimensional projection of the d -dimensional hypersurface from the set \mathcal{U}_d , respectively.

This series is reminiscent of the sequence of normal field extensions in Galois theory, reinforcing our interpretation of the UCYA as a ‘normal’ algebra. However, the fact that one should use two dual-conjugate operations, intersection $\hat{\sigma}$ and projection $\hat{\pi}$, to complete the procedure of extension is an essential new feature beyond Galois theory. For the construction of the Universal Calabi-Yau Algebra via the r -ary composition operations ω_r of the RWVs, it is not enough to use only the intersection operators, but one should also use the intersection-projection duality.

Combined with the arity structure, this duality gives very naturally the fibre structures of CY_d hypersurfaces. To see these, one needs to use a link between the two polyhedra in a mirror pair. In the holomorphic quotient formalism [38], one can easily see that the intersection of the mirror partner determines a corresponding fibre structure of CY_d . For example, one can easily see the fibre structures in $K3$ spaces, which are determined by the interplay of the integer point-monomials in the intersection, $\sigma(\Delta)$, and in the projection, $\pi(\Delta^*)$. These fibre structures are defined by the following dual relations between point-monomials in the intersection of the polyhedron and the projection in mirror polyhedron,

respectively: $\sigma(\Delta) = 4 \rightarrow \pi(\Delta^*) = 10$, $\sigma(\Delta) = 5 \rightarrow \pi(\Delta^*) = 9$, $\sigma(\Delta) = 6 \rightarrow \pi(\Delta^*) = 8$, $\sigma(\Delta) = 7 \rightarrow \pi(\Delta^*) = 7$, $\sigma(\Delta) = 8 \rightarrow \pi(\Delta^*) = 6$, $\sigma(\Delta) = 9 \rightarrow \pi(\Delta^*) = 5$, $\sigma(\Delta) = 10 \rightarrow \pi(\Delta^*) = 4$ producing planar fibres, and $\sigma(\Delta) = 3 \rightarrow \pi(\Delta^*) = 3$ producing a line-segment fibre [34].

Some of the $K3, CY_3, \dots$ chains satisfy the following condition:

$$\begin{aligned}\sigma(\Delta^n) &= \pi(\Delta^n) = \Delta^{n-1}, \\ \pi(\Delta^{n*}) &= \sigma(\Delta^{n*}) = \Delta^{n-1*},\end{aligned}\tag{48}$$

which means that the number of integer points in the intersection of the polyhedron (mirror polyhedron) forming the reflexive polyhedron of lower dimension is equal to the number of projective lines crossing these integer points of the polyhedron (mirror). The projections of these lines on a plane in the polyhedron and a plane in its mirror polyhedron reproduce, of course, the reflexive polyhedra of lower dimension. Only for self-dual polyhedra can one have

$$\sigma(\Delta^n) = \pi(\Delta^n) = \pi(\Delta^{n*}) = \sigma(\Delta^{n*}) = \Delta^{n-1} = \Delta^{n-1*},\tag{49}$$

namely the most symmetrical form of these relations.

2.5 Mirror symmetry and CICY in the Universal Calabi-Yau Algebra

We have already mentioned briefly the significance of mirror duality in the Universal Calabi-Yau Algebra (UCYA). In this subsection we stress the link between mirror symmetry and CICY spaces in the UCYA.

The normal universal composition operations defined earlier give an algebraic description of the full underlying set of reflexive weight vectors $\bigcup_r A_n^r$, where each set A_n^r contains the full list of reflexive weight vectors in each dimension n and with all internal arity structures $2 \leq r \leq n$. The hypersurfaces corresponding to the set A_n , with fixed $n = d + 2$ can be called the *single* or *zero-level* $\{CY_d\}^0$ (where in general the level $l = r - 1$), which are each described by a single polynomial equation:

$$\{CY_d\}^{(0)} = A_{d+2}\tag{50}$$

The set $\{CY_d\}^0$ with fixed d gives only a small part of the full set of CY_d hypersurfaces:

$$\{CY_d\}^0 \subset \sum_l \{CY_d\}^l\tag{51}$$

where

$$\begin{aligned}
\{CY_d\}^1 &= A_{d+3}^2, \\
\{CY_d\}^2 &= A_{d+4}^3, \\
\{CY_d\}^3 &= A_{d+5}^4, \\
\dots\dots\dots &= \dots\dots\dots
\end{aligned} \tag{52}$$

where positive integer number $l = r - 1$ characterize the level (arity) of the set. We illustrate these relations in the arity-dimension (n, r) plot Fig. 3, where the full set of CY_3 is seen to be obtained from the collection of all integer points on the line $n = r + d + 1 : d = 3$, i.e., $\{CY_3\} = A_5 + A_6^2 + A_7^3 + A_8^4 + \dots$. The integers on this plot, A_n^r , describe not only the number of the arity- r chains of $CY_{d=n-2}$ spaces, but due to the UCYA also give the $CICY$ s of fixed dimension on the different levels.

The hypersurfaces $\{CY_d\}^l$ on the following levels $l = 1, 2, 3, \dots$ are described by two, three, four, ... polynomial equations in CP^{d+2} , CP^{d+3} , CP^{d+4} , ... spaces, respectively, i.e., for the equations of a Calabi-Yau hypersurface $\{CY_d\}^l$ on the level $l = r - 1$, one should consider the r polynomial equations in CP^{d+r} complex projective space:

$$\begin{aligned}
A_{d+3}^2 \rightarrow \{CY_d\}^{(1)} &= \{\vec{x} \in CP^{d+2} \mid \wp_{i_1}(\vec{x}) = \wp_{i_2}(\vec{x}) = 0\} \\
A_{d+4}^3 \rightarrow \{CY_d\}^{(2)} &= \{\vec{x} \in CP^{d+3} \mid \wp_{i_1}(\vec{x}) = \wp_{i_2}(\vec{x}) = \wp_{i_3}(\vec{x}) = 0\} \\
A_{d+5}^3 \rightarrow \{CY_d\}^{(3)} &= \{\vec{x} \in CP^{d+4} \mid \wp_{i_1}(\vec{x}) = \wp_{i_2}(\vec{x}) = \wp_{i_3}(\vec{x}) = \wp_{i_4}(\vec{x}) = 0\} \\
\dots &= \dots,
\end{aligned} \tag{53}$$

where $\vec{x} \equiv (x_1, \dots, x_{d+r+1}) \in CP^{d+r}$. The action of the polynomial equations is equivalent to the simultaneous action of the normal sets of extended RWVs, i.e.,

$$\begin{aligned}
\{\wp_{i_1}(\vec{x}) = \wp_{i_2}(\vec{x}) = 0\} &\Leftrightarrow \left(\vec{k}_{d+3}^{(ex)(i_1)} \cap \vec{k}_{d+3}^{(ex)(i_2)} \right) \\
\{\wp_{i_1}(\vec{x}) = \wp_{i_2}(\vec{x}) = \wp_{i_3}(\vec{x}) = 0\} &\Leftrightarrow \left(\vec{k}_{d+4}^{(ex)(i_1)} \cap \vec{k}_{d+4}^{(ex)(i_2)} \cap \vec{k}_{d+4}^{(ex)(i_3)} \right) \\
\{\wp_{i_1}(\vec{x}) = \dots = \wp_{i_4}(\vec{x}) = 0\} &\Leftrightarrow \left(\vec{k}_{d+5}^{(ex)(i_1)} \cap \dots \cap \vec{k}_{d+5}^{(ex)(i_4)} \right) \\
\dots &\Leftrightarrow \dots
\end{aligned} \tag{54}$$

Thus, within the UCYA the description of the CICY is very simple. These manifolds correspond to the ‘normal’ submanifolds of the manifolds of the higher dimensions, and can very easily be described by the intersections of extended RWVs. This can be seen from the following identity:

$$\left(\vec{k}_{d+r+1}^{(ex)(i_1)} \cap \dots \cap \vec{k}_{d+r+1}^{(ex)(i_s)} \cap \dots \cap \vec{k}_{d+r+1}^{(ex)(i_r)}\right) \equiv \left(\vec{k}_{d+r+1}^{(ex)(i_1)} \cap \dots \cap \vec{k}_{d+r+1} \cap \dots \cap \vec{k}_{d+r+1}^{(ex)(i_r)}\right) \quad (55)$$

Thus, the set of hypersurfaces with fixed d corresponding to all the non-zero level numbers $l = r - 1 = 1, 2, \dots$ contains the full list of CICYs in a fixed number d of dimensions:

$$\{CICY\}_d = \sum_{l \geq 1} \{CY\}_d^l = \sum_{r \geq 2, n} \{A_{d+r+1}^r\}, \quad \text{where} \quad n = d + r + 1. \quad (56)$$

Mirror symmetry can be expressed in the UCYA as a projection-intersection symmetry between hypersurfaces and their mirror partners, respectively. For example, a d -dimensional hypersurface X_d should have in the $(d - r)$ -dimensional intersection (projection) a subspace of similar structure with lower dimension: X_{d-r} and, consequently, the mirror hypersurface X_d^* should have in the corresponding projection (intersection) the mirror subspace X_{d-r}^* .

The main conjecture of the UCYA is that it is closed under the mirror symmetry transformation, after taking into account all these r -ary composition operations. The set of lowest-level $\{CY_d\}^{(0)}$ spaces is not closed under mirror duality: only the full list of $\{CY_d\}^{(0)}$ and $CICY_d^l$ spaces, corresponding to the integer points on the line $n = d + r + 1$ of the $(r - n)$ plot is closed under this duality.

3 The Mechanisms of Normal Expansion

To understand the origin of the normal extension, i.e., to find in the set $\mathcal{A}_n^{(n-r+1)}$ ‘good’ r -extended RWVs that define a normal intersection, one must consider generally the set of r RWVs and analyze their intersections. To this end, we now formulate more precisely the correspondence defining normal ω_r composition operations on RWVs, considering first the case with arity $r = 2$. Let two fixed RWVs of dimension n , $\vec{k}_n^a = (k_1^a | k_2^a, \dots, k_n^a) [d_a] \in A_n$ and $\vec{k}_n^b = (k_1^b | k_2^b, \dots, k_n^b) [d_b] \in A_n$ have the following sets of monomials,

$$\begin{aligned} \vec{\mu}^i &= (\mu_1 | \mu_2, \dots, \mu_n)^i, \\ \vec{\nu}^j &= (\nu_1 | \nu_2, \dots, \nu_n)^j, \end{aligned} \quad (57)$$

such that $\vec{\mu}^i \cdot \vec{k}_n^a = d_a$ and $\vec{\nu}^j \cdot \vec{k}_n^b = d_b$.

We have separated by the symbol $\{|\}$ the first components from the other components of the RWVs \vec{k}_n^a and \vec{k}_n^b , and similarly in the corresponding monomials. For each of the vectors \vec{k}_n^a and \vec{k}_n^b , we consider subsets of the monomials,

$$\begin{aligned}\vec{\mu}^{i'} &= (\mu_1|\mu_2, \dots, \mu_n)^{i'} \subset S_n^a, \\ \vec{\nu}^{j'} &= (\nu_1|\nu_2, \dots, \nu_n)^{j'} \subset S_n^b,\end{aligned}\tag{58}$$

with the following two properties:

$$\mu_2 = \nu_2, \quad \mu_3 = \nu_3, \quad \dots, \quad \mu_n = \nu_n\tag{59}$$

or

$$S_n^a|_{x_1=0} = S_n^b|_{x_1=0} = S_{n-1};\tag{60}$$

and, moreover, the set of vectors

$$S_{n-1} : (\vec{\mu}_{red})^i = (\mu_2, \dots, \mu_n)^i\tag{61}$$

corresponds in the integer lattice to a ‘reduced’ reflexive polyhedron of dimension $(n-1)$.

For now, we note that the weight vectors \vec{k}_n^a and \vec{k}_n^b also determine in the integer lattice the reduced $(n-1)$ -dimensional reflexive polyhedron. Thus, when we consider the extensions of these two RWVs:

$$\begin{aligned}\vec{k}_{n+1}^{(ex)a} &= (0, k_1|k_2, \dots, k_n)^a, \\ \vec{k}_{n+1}^{(ex)b} &= (k_1, 0|k_2, \dots, k_n)^b,\end{aligned}\tag{62}$$

and, correspondingly, their common monomials which can be obtained from the extensions of the set S_{n-1} , i.e.,

$$(\vec{\mu}^{ex})^i = S_{n+1}^{ex} = (\nu_1, \mu_1|\mu_2 = \nu_2, \dots, \mu_n = \nu_n)^i\tag{63}$$

which satisfy the following two conditions:

$$\begin{aligned}\vec{k}_{n+1}^{(ex)a} \cdot (\vec{\mu}^{ex})^i &= [d_a] \\ \vec{k}_{n+1}^{(ex)b} \cdot (\vec{\mu}^{ex})^i &= [d_b],\end{aligned}\tag{64}$$

again the set of extended monomials S_{n+1}^{ex} produces in the integer $n+1$ -dimensional lattice the reduced $(n-1)$ -dimensional reflexive polyhedron.

One can now see that the weight vector $\vec{k}_{n+1}[d] = \vec{k}_{n+1}^{(ex)a} + \vec{k}_{n+1}^{(ex)b}$ satisfies the condition:

$$\vec{k}_{n+1} \cdot (\vec{\mu}^{ex})^i = [d] = [d_a] + [d_b], \quad (65)$$

what means that the n -dimensional polyhedron corresponding to this new weight vector contains itself the reduced $(n-1)$ -dimensional reflexive polyhedron. We illustrate how this works for arity 2, the expansion giving the condition that the central point-monomial is inside new polyhedron. For this one can consider in the reduced reflexive polyhedron an expanded point $P_0 = (\nu_1^0, \mu_1^0 | \mu_2, \dots, \mu_n)$ and check that its right and left neighbour-points, $P_{+,-} = (\nu_1^0 + (-)\xi^+, \mu_1^0 - (+)\xi^- | \mu_2, \dots, \mu_n)$, satisfy the condition $\vec{k}_n \cdot P_{+,-} = d$. One can easily check this, for example, when the first expanded components of the n -dimensional reflexive weight vectors, \vec{k}_n^a and \vec{k}_n^b , are both units, i.e., $k_1^a = k_2^b = 1$. It is easy to see that each n -dimensional reflexive weight vector has at least n normal extensions into $(n+1)$ -dimensional RWVs ².

The generalization to other cases: RWVs in different dimensions, ‘good’ triples, etc. ... can be done in a similar way. Thus, we can reformulate the conjecture of a normal arity extension, for any arity, in the language of RWVs. As an immediate consequence, we can get chains in higher dimensions. For example, each $RFW(n-1)$ can give us $(n-1)$ chains of dimension n , and it is very easy to understand the origin of n -dimensional chains in terms of $(n-1)$ -dimensional chains. This gives us a very convenient rule for getting arity-2 n -dimensional chains, by first taking into account the eldest vectors in $(n-1), (n-2), \dots$ dimensions.

We are now ready to demonstrate more explicitly how our construction acts in the arity-dimension plots, Fig. 2 and Fig. 3. In so doing, we also show how this construction makes explicit the structural systematics of the singularities of CY_d -folds. Our method in this section is based on normal expansions of RWVs, which are related to Batyrev’s reflexive polyhedra, but we shall also use some results obtained by the Diophantine decomposition of IMs.

3.1 Analysis of A_r and D_r lines

Starting from dimension $n=1$, where we have $A_1^1 = \vec{k}_1^{eld} = (1)$, the extension to $n=2$ is immediate, and it is trivial to see that there is just chain of arity 2:

$$\omega_2 : \mathbb{A}_2^{(2)} \star \mathbb{A}_2^{(2)} \mapsto A_2^2, \quad (66)$$

²Also, one can check that if some extension of the eldest vector is good, then all vectors in the chain will also have analogous good extensions.

where the set $\mathcal{A}_2^{(2)}$ contains only two extended vectors: $\vec{k}_1^{1(ex)} = (0, 1)$ and $\vec{k}_1^{2(ex)} = (1, 0)$. There is one ‘good’ intersection:

$$\vec{k}_1^{1(ex)} \cap \vec{k}_1^{2(ex)} = (1)', \quad (67)$$

where the symbol $1'$ signifies that this intersection has only point, corresponding to the unit monomial $E_1 = (1, 1)$, for which the algebra gives just one eldest RWV:

$$\vec{k}_1^{1(ex)} \cup \vec{k}_1^{2(ex)} = \vec{k}_2 = (1, 1). \quad (68)$$

Thus $A_2^1 = A_2^2 = \vec{k}_2 = (1, 1)$. This vector corresponds to a trivial reflexive polyhedron that just consists of a line segment, and to the three monomials:

$$C_2 = (0, 2), \quad E_2 = (1, 1), \quad C_2 = (2, 0). \quad (69)$$

Following the diagonal red A_r line to higher dimensions $n \geq 3$, in each case we also get only one chain of arity $\omega_r = n$, with a single eldest vector: $\vec{k}_n^{eld} = (1, \dots, 1)$. Thus, every chain with the maximal arity n is determined by just the one unit RWV $\vec{k}_1 = (1)$, making the following set of extensions to higher-dimensional vectors:

$$\begin{aligned} \vec{k}_n^{1(ex)} &= (1, 0, 0, \dots, 0, 0, 0), \\ \vec{k}_n^{2(ex)} &= (0, 1, 0, \dots, 0, 0, 0), \\ \dots &= \dots\dots\dots \\ \vec{k}_n^{(n-1)(ex)} &= (0, 0, 0, \dots, 0, 1, 0), \\ \vec{k}_n^{n(ex)} &= (0, 0, 0, \dots, 0, 0, 1), \end{aligned} \quad (70)$$

This set of extended vectors has only one ‘good’ intersection:

$$\bigcup_{i=1}^{i=n} \vec{k}_n^{i(ex)} = 1, \quad (71)$$

corresponding to the existence of just one chain with the eldest vector $\vec{k}_n^{eld} = (1, \dots, 1)$. The number of points in the intersection with the integer Z_n lattice defining the corresponding polyhedron is given by the binomial coefficient C_{2n-1}^n .

In the $K3$ case, for example, the only chain of arity 4 has as eldest vector $\vec{k}_4^{eld} = (1, 1, 1, 1)$, i.e.:

$$\omega_4 : \underbrace{\mathcal{A}_4^{(1)} \star \dots \star \mathcal{A}_4^{(1)}}_4 \mapsto A_4^4, \quad (72)$$

where $\mathbb{A}_4^{(1)}$ contains 4 extended vectors. To obtain the complete listing of $K3$ spaces in this chain, one can use mirror symmetry to find the maximal values of the integer parameters m describing the chain, or use the maximal expansion of the unit monomial $E_4 = (1, 1, 1, 1)$ in terms of four monomials P_1, \dots, P_4 with the following condition: $1/4(P_1 + \dots + P_4) = (1, 1, 1, 1)$.

We introduce the following notation for the Diophantine decomposition of any monomial: $D_n[p_1, p_2, \dots, p_i/d]$ with $2 \leq i \leq n$, n is the dimension and p_1, \dots, p_i are integers with $d = p_1 + \dots + p_i$, as defined by the following equation:

$$D_n[p_1, p_2, \dots, p_i/d_D] \cdot \vec{\mu}_0 = (p_1 P_1 + p_2 P_2 + \dots + p_i P_i)/d, \quad (73)$$

where $\vec{\mu}_0$ is a primordial monomial of dimension n and $\vec{\mu}_1, \dots, \vec{\mu}_i$ are the expanded monomials. According to this notation, the above Diophantine decomposition can be represented in the form $D_4[1, 1, 1, 1/4] \cdot E_4 = \{P_1, P_2, P_3, P_4\}$ with the usual Diophantine property: $(P_1 + \dots + P_4)/4 = E_4$, in this case, $p_1 = \dots = p_4 = 1$ and $d = 4$. The definition of an invariant monomial (IM) is very closely connected with this definition of Diophantine expansion. We call the monomials participating in such a Diophantine expansion (73) a *set of IMs*. The unit monomial E_n is an IM by definition, then with the decomposition $D_n[1, 1/2]$ we produce a triple of IMs, E_n, C_1, C_2 obeying the condition $(C_1 + C_2) = E_n$, etc..

This expansion has just 42 different irreducible solutions, which is much less than the complete list of $K3$ RWVs. This example re-emphasizes that one cannot get full information about Calabi-Yau d -folds by following only the first diagonal line on the arity-dimension plot. Fortunately, this also means that the internal structures of $K3, CY_3, \dots$ d -folds are much more complicated and interesting.

The next example is the CY_3 case with $n = 5$, where there is again just one chain with eldest vector $k_5^{(eld)} = (1, 1, 1, 1, 1)$ that can be obtained with arity 5. The listing of this chain again can again be obtained using mirror symmetry or by the maximal expansion of the unit monomial $E_5 = (1, 1, 1, 1, 1)$ in terms of five monomials P_1, \dots, P_5 . The corresponding Diophantine equation, $E_5 = 1/5(P_1 + P_2 + \dots + P_5)$, has 7269 solutions, giving the number of RWVs in $A_5^{(5)}$, again much smaller than the total number of CY_3 spaces. *A fortiori*, the same is true along this line for all dimensions.

These simple examples show that, although each CY_d -fold must contain a central point, one cannot obtain all the d -folds simply by expanding the unit monomial. The geometrical reason why the expansion of the unit monomial E_n as the sum of n monomials cannot give the full list of CY_d -folds lies in the complex composite structure of its geometry. The geometry of CY_d -folds has a composite ‘Russian doll’ structure. In the intersection by the $(d - 1)$ -hyperplane through the centre of a d -fold, one can find a $(d - 1)$ -fold, then in the

next intersection by the $(d - 2)$ -hyperplane one can find a $(d - 2)$ -fold, etc.. The picture is completed by mirror symmetry and the intersection-projection symmetry of CY_d -folds. We note that, according to the arity-dimension structure, if an intersection by a hyperplane does not contain a d -fold, then the corresponding d -folds have to be seen by projection on this hyperplane.

Only if we know all the ω_r operations with which a given vector can be obtained can we know all the possible sets of expansions of this reflexive vector, corresponding to the different chains in which it appears in different dimensions, as illustrated in Figs 3, 4, 5. To get new information on the structures of CY_d -folds, we should study CY_d -folds obtained by operations corresponding to the points on the second sloping line. This $A_n^{(n-1)}$ set of CY_d -folds has a somewhat more complicated structure and, according to our approach, is described by the weight vectors (1) and (1,1), extended to dimension n . This already implies that the internal geometrical structures of these CY_d folds is connected algebraically with the properties of conic monomials.

The two examples shown in Figs. 4 and 5 display different expansions of RWVs that illustrate how the arity structure of the Universal Calabi-Yau Algebra reveals the full algebraic structure of all RWVs. It gives a complete algebraic expansion of all RWVs via the sets of extended RWVs constructed already from RWVs of lower dimension.

There appear constructions with two different arities r already in three dimensions: apart from the maximal arity 3, also the lower arity 2 construction can give some information about the composite structure of the RWVs and the corresponding polyhedra. The points on the second (green) diagonal line in the Figures start with this case. In dimension three, it is easy to construct the complete list of RWVs in the arity-3 chain whose eldest vector is $A_3^{(1)} = \vec{k}_3^{eld} = (1, 1, 1)$, using the expansion technique or mirror symmetry: they are $(1, 1, 1), (1, 1, 2), (1, 2, 3)$. Analyzing the decomposition of the RWVs in chains with arity 2, we find these three vectors in two different chains of arity 2:

$$\omega_2 : \mathbb{A}_3^{(2)} \star \mathbb{A}_3^{(2)} \mapsto \mathbb{A}_3^2, \quad (74)$$

with two different eldest RWVs. To generate these two chains, one should consider two RWVs of dimensions one and two, and extend them in all possible ways to dimension three, i.e.:

$$\begin{aligned} \vec{k}_1^{i(ex)} &= (0, 0, 1) & + & & 3 \text{ permutations} \\ \vec{k}_2^{i(ex)} &= (0, 1, 1) & + & & 3 \text{ permutations} \end{aligned} \quad (75)$$

Then, in the set of six extended vectors, one should consider all C_6^2 possible cases, so as to find the only ‘good’ triple intersections, corresponding to the arity two construction. The two chains of arity two are the following:

$$\vec{k}_1^{1(ex)} = (1, 0, 0), \quad \vec{k}_2^{2(ex)} = (0, 1, 1) \quad (76)$$

and

$$\vec{k}_2^{1(ex)} = (0, 1, 1), \quad \vec{k}_2^{2(ex)} = (1, 0, 1) \quad (77)$$

with the eldest vectors being, respectively:

$$\vec{k}^{(eld)} = (1, 1, 1) \quad (78)$$

and

$$\vec{k}^{(eld)} = (1, 1, 2). \quad (79)$$

The two different good intersections giving us line-segment reflexive polyhedra can be described by the following sets of linear and conic monomials:

$$C_1 = (1, 2, 0), \quad C_2 = (1, 0, 2), \quad E_3 = (1, 1, 1) \quad (80)$$

and

$$C_1 = (2, 2, 0), \quad C_2 = (0, 0, 2), \quad E_3 = (1, 1, 1) \quad (81)$$

with the following universal property: $C_1 + C_2 = 2 \cdot E_3$.

In the $K3$ case in four dimensions, there are already four chains of arity 3 on the second diagonal D_r line:

$$\omega_3 : \mathbb{A}_4^{(2)} \star \mathbb{A}_4^{(2)} \star \mathbb{A}_4^{(2)} \mapsto A_4^3, \quad (82)$$

with four corresponding eldest RWVs. To get these four chains, one can consider two RWVs of dimension one and two, and extend them in all possible ways to dimension four, i.e.:

$$\begin{aligned} \vec{k}_1^{ex} &= (0, 0, 0, 1) & + & & 4 \text{ permutations} \\ \vec{k}_2^{ex} &= (0, 0, 1, 1) & + & & 6 \text{ permutations} \end{aligned} \quad (83)$$

Now, within the set of ten extended vectors, one should look for ‘good’ triple intersections corresponding to the arity 3, among the C_{10}^3 possible cases. The four chains of arity 3 found in this way are the following:

$$\begin{aligned}
\vec{k}^{i(ex)} &= (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 1, 0), (1, 0, 0, 1), (0, 1, 0, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1) \\
\vec{k}^{i(ex)} &= (1, 0, 0, 1), (0, 1, 0, 1), (1, 0, 1, 0)
\end{aligned} \tag{84}$$

with the following eldest vectors:

$$\begin{aligned}
\vec{k}^{(eld)} &= (1, 1, 1, 1) \\
\vec{k}^{(eld)} &= (1, 1, 1, 2) \\
\vec{k}^{(eld)} &= (1, 1, 1, 3) \\
\vec{k}^{(eld)} &= (2, 1, 1, 2)
\end{aligned} \tag{85}$$

These four different intersections give us segment reflexive polyhedra that can be described by the following sets of conic monomials:

$$\begin{aligned}
C_1 &= (1, 1, 2, 0), & C_2 &= (1, 1, 0, 2), \\
C_1 &= (2, 2, 1, 0), & C_2 &= (0, 0, 1, 2), \\
C_1 &= (2, 2, 2, 0), & C_2 &= (0, 0, 0, 2), \\
C_1 &= (2, 2, 0, 0), & C_2 &= (0, 0, 2, 2),
\end{aligned} \tag{86}$$

with the following universal property: $C_1 + C_2 = 2 \cdot E_4$. To list these four chains, one can again use mirror symmetry, or one may consider the decompositions of the following conic monomials:

$$\begin{aligned}
C_1 &= (2, 2, 2, 0) \rightarrow P_1, P_2, P_3 : 1/3(P_1 + P_2 + P_3) = C_1, & C_2 &= (0, 0, 0, 2); \\
C_1 &= (2, 2, 1, 0) \rightarrow P_1, P_2, P_3 : 1/3(P_1 + P_2 + P_3) = C_1, & C_2 &= (0, 0, 1, 2); \\
C_1 &= (2, 1, 1, 0) \rightarrow P_1, P_2, P_3 : 1/3(P_1 + P_2 + P_3) = C_1, & C_2 &= (0, 1, 1, 2); \\
C_1 &= (2, 2, 0, 0) \rightarrow P_1, P_2 : 1/2(P_1 + P_2) = C_1 \\
C_2 &= (0, 0, 2, 2) \rightarrow P_3, P_4 : 1/2(P_3 + P_4) = C_2.
\end{aligned} \tag{87}$$

For example, as was already shown in [35], there are 34 vectors in the first chain (III). The eldest and youngest vectors of this chain are $k_4^{eld} = (1, 1, 1, 3)$ and $k_4^{eld} = (7, 8, 10, 25)$, respectively. The full number of RWVs in $A_n^{(3)}$ with conic structures is 90. Thus, to get the full list of $K3$ with new conic structures, one can use a composite expansion technique, starting from the unit monomial $E_4 = (1, 1, 1, 1)$. In the first step, one can use the expansion of unit monomials to produce the conics C_1 and C_2 , giving rise to the eldest vectors and chains with arity 3. Secondly, to get the complete listings of the chains, one should construct the decompositions of the conics in terms of three monomials, as found by solving the corresponding Diophantine equations.

Similarly, in the CY_3 case there are already six chains of arity 4:

$$\omega_4 : \underbrace{\mathbb{A}_5^{(2)} \star \dots \star \mathbb{A}_5^{(2)}}_4 \mapsto A_5^4, \quad (88)$$

They can be obtained by extending the two simple RWVs (1) and (1,1) to dimension five. The number of distinct extended vectors is now $15 = 5 + 10$. Then one should look for ‘good’ quadruple intersections among the C_{15}^4 possible combinations. The resulting six chains of arity 4 are the following:

$$\begin{aligned} \vec{k}^{i(ex)} &= (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 0, 1, 1) \\ \vec{k}^{i(ex)} &= (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (1, 0, 0, 0, 1), (0, 1, 0, 0, 1) \\ \vec{k}^{i(ex)} &= (1, 0, 0, 0, 0), (0, 0, 0, 1, 1), (0, 0, 1, 0, 1), (0, 1, 0, 0, 1) \\ \vec{k}^{i(ex)} &= (0, 0, 0, 1, 1), (0, 0, 1, 0, 1), (0, 1, 0, 0, 1), (1, 0, 0, 0, 1) \\ \vec{k}^{i(ex)} &= (0, 0, 1, 0, 0), (0, 1, 0, 1, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 1) \\ \vec{k}^{i(ex)} &= (1, 0, 0, 1, 0), (0, 1, 0, 1, 0), (0, 0, 1, 1, 0), (1, 0, 0, 0, 1) \end{aligned} \quad (89)$$

with the following six eldest vectors:

$$\begin{aligned} \vec{k}^{(eld)} &= (1, 1, 1, 1, 1) \\ \vec{k}^{(eld)} &= (1, 1, 1, 1, 2) \\ \vec{k}^{(eld)} &= (1, 1, 1, 1, 3) \\ \vec{k}^{(eld)} &= (1, 1, 1, 1, 4) \\ \vec{k}^{(eld)} &= (1, 2, 1, 2, 1) \\ \vec{k}^{(eld)} &= (2, 1, 1, 3, 1) \end{aligned} \quad (90)$$

The intersections giving the segment reflexive polyhedra can be described by the following sets of conic monomials:

$$\begin{aligned}
C_1 &= (1, 1, 1, 2, 0), & C_2 &= (1, 1, 1, 0, 2), \\
C_1 &= (2, 2, 1, 1, 0), & C_2 &= (0, 0, 1, 1, 2), \\
C_1 &= (1, 2, 2, 2, 0), & C_2 &= (1, 0, 0, 0, 2), \\
C_1 &= (2, 2, 2, 2, 0), & C_2 &= (0, 0, 0, 0, 2), \\
C_1 &= (2, 2, 1, 0, 0), & C_2 &= (0, 0, 1, 2, 2), \\
C_1 &= (2, 2, 2, 0, 0), & C_2 &= (0, 0, 0, 2, 2),
\end{aligned} \tag{91}$$

with the following Diophantine property: $C_1 + C_2 = 2 \cdot E_5$.

To find the complete list of 14,017 RWVs in the first chain, one can consider the following expansions of the monomial C_1 (C_2):

$$C_1 = (2, 2, 2, 2, 0) \rightarrow P_1, P_2, P_3, P_4 : 1/4(P_1 + P_2 + P_3 + P_4) = C_1, \quad C_2 = (0, 0, 0, 0, 2). \tag{92}$$

The eldest and the youngest vectors of this chain are $k_5^{eld} = (1, 1, 1, 1, 4)[8]$ and $k_5^{yng} = (75, 84, 86, 98, 343)[686]$, respectively.

So far we have mainly been recapitulating results for lower-dimensional CY_d -folds, explaining them in a manner suitable for extensions to higher dimensions, and now we give some first results for the CY_4 case. It is easy to see that there are already 9 chains of arity 5:

$$\omega_5 : \underbrace{\mathbb{A}_6^{(2)} \star \dots \star \mathbb{A}_6^{(2)}}_5 \mapsto A_6^5, \tag{93}$$

These can be obtained by extending the two RWVs (1) and (1,1) to dimension 6, in all $21 = 6 + 15$ different ways. There are C_{21}^5 different combinations to search for ‘good’ quadruple intersections, among which we find the following 9 chains of arity 5:

$$\begin{aligned}
\vec{k}^{i(ex)} &= (1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 1, 0, 0, 0, 1), (1, 0, 0, 0, 0, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 0, 1), (1, 0, 0, 0, 0, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 0, 0, 1, 0), (0, 0, 0, 1, 0, 1), (0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 0, 1), (1, 0, 0, 0, 0, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 0, 0, 1, 1), (0, 0, 0, 1, 0, 1), (0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 0, 1), (1, 0, 0, 0, 0, 1)
\end{aligned}$$

$$\begin{aligned}
\vec{k}^{i(ex)} &= (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (1, 0, 0, 0, 1, 0), (1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 1) \\
\vec{k}^{i(ex)} &= (0, 0, 0, 1, 0, 0), (1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 0, 1), (1, 0, 0, 0, 1, 0) \\
\vec{k}^{i(ex)} &= (1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 1, 0), (0, 0, 1, 1, 0, 0) \\
\vec{k}^{i(ex)} &= (1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 0, 1), (0, 0, 0, 1, 0, 1), (0, 0, 0, 1, 1, 0)
\end{aligned} \tag{94}$$

with the following 9 eldest vectors:

$$\begin{aligned}
\vec{k}^{(eld)} &= (1, 1, 1, 1, 1, 1) \\
\vec{k}^{(eld)} &= (1, 1, 1, 1, 1, 2) \\
\vec{k}^{(eld)} &= (1, 1, 1, 1, 1, 3) \\
\vec{k}^{(eld)} &= (1, 1, 1, 1, 1, 4) \\
\vec{k}^{(eld)} &= (1, 1, 1, 1, 1, 5) \\
\vec{k}^{(eld)} &= (2, 1, 1, 1, 1, 2) \\
\vec{k}^{(eld)} &= (2, 1, 1, 1, 1, 3) \\
\vec{k}^{(eld)} &= (1, 2, 2, 1, 1, 3) \\
\vec{k}^{(eld)} &= (1, 1, 1, 2, 1, 4)
\end{aligned} \tag{95}$$

These 9 different ‘good’ intersections yield segment reflexive polyhedra described by the following sets of conic monomials:

$$\begin{aligned}
C_1 &= (1, 1, 1, 1, 2, 0), & C_2 &= (1, 1, 1, 1, 0, 2), \\
C_1 &= (2, 2, 1, 1, 1, 0), & C_2 &= (0, 0, 1, 1, 1, 2), \\
C_1 &= (2, 2, 2, 1, 1, 0), & C_2 &= (0, 0, 0, 1, 1, 2), \\
C_1 &= (2, 2, 2, 2, 1, 0), & C_2 &= (0, 0, 0, 0, 1, 2), \\
C_1 &= (2, 2, 2, 2, 2, 0), & C_2 &= (0, 0, 0, 0, 0, 2), \\
C_1 &= (2, 2, 1, 1, 0, 0), & C_2 &= (0, 0, 1, 1, 2, 2), \\
C_1 &= (2, 2, 2, 1, 0, 0), & C_2 &= (0, 0, 0, 1, 2, 2), \\
C_1 &= (2, 2, 2, 0, 0, 0), & C_2 &= (0, 0, 0, 2, 2, 2), \\
C_1 &= (2, 2, 2, 2, 0, 0), & C_2 &= (0, 0, 0, 0, 2, 2),
\end{aligned} \tag{96}$$

with the following Diophantine property: $C_1 + C_2 = 2 \cdot E_5$.

Continuing along this line to the highest arity in each higher dimension, we find the following numbers of different eldest vectors and chains: 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100, 110, 121, 132, 144, ..., which are given by a simple recurrence relation, as we show in the next section.

3.2 Analysis of the E_r line

We now consider Calabi-Yau d -folds with more complicated structures, corresponding to the sets $A_n^{(n-2)}$ on the arity-dimension plot. These can be constructed from RWVs of lower dimensions: (1); (1,1); (1,1,1),(1,1,2),(1,2,3), suitably extended to dimension n .

3.2.1 The $K3$ Case

The E_r line in this case corresponds to arity 2:

$$\omega_2 : \mathbb{A}_4^{(3)} \star \mathbb{A}_4^{(3)} \mapsto A_4^2, \quad (97)$$

where $\mathbb{A}_4^{(2)}$ contains $50 = 4 + 6 + 6 + 12 + 24$ extended vectors. We find 22 eldest vectors and chains of arity 2, corresponding to $K3$ spaces with a composite structure containing a $d = 1$ complex curve. To find the full lists of RWVs in these 22 chains, one may use the argument that the maximal values of integer parameters m are bounded by the dimensions of the corresponding extended vectors, as was done in [34].

The composite structure of d -folds along this third diagonal line on the arity-dimension plot is determined by cubic and quartic monomials, of which we now give some examples. Chain I in Table 1 is determined by the ‘good’ intersection of two extended vectors

$$\vec{k}_1^{(ex)} = (1, 0, 0, 0), \quad \vec{k}_3^{(ex)} = (0, 1, 1, 1). \quad (98)$$

This intersection contains 10 monomial points, producing a plane reflexive polyhedron, corresponding to the RWV $\vec{k}_3 = (1, 1, 1)$ [34]. These 10 monomials include the following three cubic monomials:

$$P_1 = (1, 3, 0, 0), \quad P_2 = (1, 0, 3, 0), \quad P_3 = (1, 0, 0, 3), \quad (99)$$

with the Diophantine property:

$$\frac{1}{3}(P_1 + P_2 + P_3) = E_4 = (1, 1, 1, 1). \quad (100)$$

Among the total of 22 chains there are 20 chains with the same property. In turn, among these 20 cases, the intersection for each of these chains contains itself some triple monomials, P_1, P_2, P_3 , of which at least one is cubic, i.e., has degree 3.

Only chains VIII, IX and XX have no such triples. However, the intersections of the chains VIII and IX have another Diophantine property, as they are determined by the intersection of the following pairs of extended vectors:

$$\begin{aligned}\vec{k}_3^{1(ex)} &= (0, 2|1, 1) & \vec{k}_3^{2(ex)} &= (1, 0|1, 2) \\ \vec{k}_3^{1(ex)} &= (1, 0|1, 2) & \vec{k}_3^{2(ex)} &= (0, 2|1, 3)\end{aligned}\tag{101}$$

For these intersections, which contain 5 monomial points, one can see in both cases the following pairs of quartic and conic monomials, respectively:

$$\begin{aligned}P_1 &= (0, 0, 4, 0), P_2 = (4, 2, 0, 0), & C_1 &= (2, 1, 2, 0), C_2 = (0, 1, 0, 2), \\ P_1 &= (4, 3, 0, 0), P_2 = (0, 1, 4, 0), & C_1 &= (2, 2, 2, 0), C_1 = (2, 2, 2, 0),\end{aligned}\tag{102}$$

with the following common Diophantine property in both cases:

$$\frac{1}{2}(P_1 + P_2) = C_1, \quad \frac{1}{2}(C_1 + C_2) = E_4.\tag{103}$$

These relations can be rewritten in the form:

$$\frac{1}{4}(P_1 + P_2 + 2 \cdot C_2) = E_4.\tag{104}$$

In the full list of 22 chains there are 20 cases with such cubic and quartic Diophantine properties.

However, the chain X is not included in these two sets. It is determined by the following intersection:

$$\vec{k}_2^{(ex)} = (0, 0, |1, 1) \quad \vec{k}_2^{(ex)} = (1, 1, |0, 0)\tag{105}$$

which has among its 9 monomial points two conic pairs:

$$\begin{aligned}C_1 &= (2, 0, 2, 0), & C_2 &= (0, 2, 0, 2), \\ \tilde{C}_1 &= (2, 0, 0, 2), & \tilde{C}_2 &= (0, 2, 2, 0)\end{aligned}\tag{106}$$

There are 17 such cases among all the 22 chains.

These examples show how different chains can be identified by decomposing IMs in terms of conic, cubic, quartic, etc., monomials. In the next section we use these Diophantine properties to study the possible structures of CY_d -folds in arbitrary dimensions. As we show there in more detail, the use of such invariant monomials provides an alternative to Batyrev's reflexive polyhedron way of constructing CY_d spaces.

The structure of the projective \vec{k}_4 vectors in the 22 chains reveals interesting interrelations between the five classical regular dual polyhedron pairs in three-dimensional space (the one-dimensional point, two-dimensional line-segment and three dimensional tetrahedron, octahedron-cube and icosahedron-dodecahedron) and Coxeter-Dynkin diagrams CD for the five types of Lie algebras: $A, D, E_{6,7,8}$, as illustrated in Fig. 1. The structures of the vectors \vec{k}_4 in the $CD_{\sigma,\pi}$ diagrams, which can be seen in the corresponding polyhedra of projective vectors, follow completely follow those of the only possible five 'extended' vectors. We show in Table 1 the A, D, E structures and the CD_{σ} diagrams of all the eldest double $K3$ projective vectors from the 22 chains [34]. In the first column, the Roman numbers enumerate the 22 chains, and we indicate by * the cases where the planar polyhedron in the intersection coincides with the polyhedron in the projection, i.e., $\sigma = \pi$.

3.2.2 The CY_3 Case

One can make similar considerations in the next dimension $n = 5$, i.e., the case of CY_3 spaces with arity 3 structure. As indicated in Fig. 2, one must first extend the five RWVs: $(1); (1, 1); (1, 1, 1), (1, 1, 2), (1, 2, 3)$ to dimension 5, and then find all the normal intersections corresponding to arity 3:

$$\omega_3 : \mathbb{A}_5^{(3)} \star \mathbb{A}_5^{(3)} \star \mathbb{A}_5^{(3)} \mapsto A_5^3, \quad (107)$$

where $\mathbb{A}_5^{(3)}$ contains $115 = 5 + 10 + 10 + 30 + 60$ extended vectors. A search of the normal triple intersections gave 259 candidates [35], of which 11 intersections have false eldest vectors that cannot be obtained consistently by an arity-2 construction. Among these 259 chains, 161 are irreducible, and 7 of these have false eldest vectors with $m = n = l = 1$.

We do not discuss these here in further detail, except to mention the biggest irreducible chain with an arity-3 structure, which consists of the RWVs $(m, n, k, 2(m + n + k), 3(m + n + k))[6(m + n + k)]$, all of which have elliptic Weierstrass fibre bundles. The total number of CY_3 hypersurfaces in this chain is 20,796, and its eldest and youngest RWVs are $(1,1,1,6,9)[18]$ and $(91,96,102,578,867)[1734]$, respectively. Its listing can be obtained by

Table 1: *The list of the eldest vectors for all 22 K3 chains: $m = n = 1$, enumerating the corresponding Coxeter-Dynkin CD_σ diagrams.*

| N | $\vec{k}_i(\text{eldest})$ | Structure | $\max \Delta(\sigma)$ | CD_σ |
|-----------|----------------------------|---|---|--|
| I | $(1, 1, 1, 1)[4]$ | $(0, 1, 1, 1)_{e6} + (1, 0, 0, 0)_a$ | $35 = \mathbf{10}_{e6} + (\sigma = \{10\}_\Delta) + \mathbf{15}_a$ | $E_6^{(1)} \leftrightarrow A_{12}^{(1)}$ |
| II^* | $(1, 1, 2, 2)[6]$ | $(0, 1, 1, 1)_{e6} + (1, 0, 1, 1)_{e6}$ | $30 = \mathbf{10}_{e6} + (\sigma = \{10\}_\Delta) + \mathbf{10}_{e6}$ | $E_6^{(1)} \leftrightarrow E_6^{(1)}$ |
| III | $(3, 1, 2, 3)[9]$ | $(0, 1, 1, 1)_{e6} + (3, 0, 1, 2)_{e8}$ | $23 = \mathbf{4}_{e6} + (\sigma = \{4\}_\Delta) + \mathbf{15}_{e8}$ | $G_2^{(1)} \leftrightarrow E_8^{(1)}$ |
| IV | $(1, 1, 1, 2)[5]$ | $(0, 1, 1, 2)_{e7} + (1, 0, 0, 0)_a$ | $34 = \mathbf{13}_{e7} + (\sigma = \{9\}_\Delta) + \mathbf{12}_a$ | $E_7^{(1)} \leftrightarrow A_9^{(1)}$ |
| V | $(1, 1, 1, 3)[6]$ | $(0, 1, 1, 2)_{e7} + (1, 0, 0, 1)_d$ | $39 = \mathbf{13}_{e7} + (\sigma = \{9\}_\Delta) + \mathbf{17}_d$ | $E_7^{(1)} \leftrightarrow D_{10}^{(1)}$ |
| VI^* | $(1, 1, 2, 4)[8]$ | $(0, 1, 1, 2)_{e7} + (1, 0, 1, 2)_{e7}$ | $35 = \mathbf{13}_{e7} + (\sigma = \{9\}_\Delta) + \mathbf{13}_{e7}$ | $E_7^{(1)} \leftrightarrow E_7^{(1)}$ |
| VII | $(1, 1, 1, 1)[4]$ | $(0, 2, 1, 1)_{e7} + (2, 0, 1, 1)_{e7}$ | $35 = \mathbf{13}_{e7} + (\sigma = \{9\}_\Delta) + \mathbf{13}_{e7}$ | $E_7^{(1)} \leftrightarrow E_7^{(1)}$ |
| $VIII$ | $(1, 2, 2, 3)[8]$ | $(1, 0, 1, 2)_{e7} + (0, 2, 1, 1)_{e7}$ | $24 = \mathbf{12}_{e7} + (\sigma = \{5\}_\Delta) + \mathbf{7}_{e7}$ | $E_7^{(1)} \leftrightarrow F_4^{(1)}$ |
| IX | $(1, 2, 1, 5)[10]$ | $(1, 0, 1, 2)_{e7} + (0, 2, 1, 3)_{e8}$ | $28 = \mathbf{7}_{e7} + (\sigma = \{5\}_\Delta) + \mathbf{16}_{e8}$ | $F_4^{(1)} \leftrightarrow E_8^{(1)}$ |
| X | $(1, 1, 1, 1)[4]$ | $(0, 0, 1, 1)_d + (1, 1, 0, 0)_d$ | $35 = \mathbf{13}_d + (\sigma = \{9\}_\square) + \mathbf{13}_d$ | $D_8^{(1)} \leftrightarrow D_8^{(1)}$ |
| XI | $(1, 1, 1, 2)[5]$ | $(1, 0, 0, 1)_d + (0, 1, 1, 1)_{e6}$ | $34 = \mathbf{15}_d + (\sigma = \{9\}_\square) + \mathbf{10}_{e6}$ | $D_8^{(1)} \leftrightarrow E_6^{(1)}$ |
| XII^* | $(1, 1, 1, 1)[4]$ | $(0, 3, 1, 2)_{e8} + (3, 0, 2, 1)_{e8}$ | $13 = \mathbf{4}_{e8} + (\sigma = \{5\}_\square) + \mathbf{4}_{e8}$ | $G_2^{(1)} \leftrightarrow G_2^{(1)}$ |
| $XIII^*$ | $(1, 1, 2, 3)[7]$ | $(0, 1, 1, 1)_{e6} + (1, 0, 1, 2)_{e7}$ | $31 = \mathbf{10}_{e6} + (\sigma = \{8\}_\square) + \mathbf{13}_{e7}$ | $E_6^{(1)} \leftrightarrow E_7^{(1)}$ |
| XIV^* | $(1, 1, 1, 2)[5]$ | $(0, 2, 1, 1)_{e6} + (2, 0, 1, 3)_{e8}$ | $18 = \mathbf{7}_{e6} + (\sigma = \{6\}_\square) + \mathbf{5}_{e8}$ | $F_4^{(1)} \leftrightarrow G_2^{(1)}$ |
| $XXII^*$ | $(1, 1, 2, 2)[6]$ | $(0, 1, 1, 2)_{e7} + (1, 0, 1, 0)_d$ | $30 = \mathbf{13}_{e7} + (\sigma = \{7\}_\square) + \mathbf{10}_d$ | $E_7^{(1)} \leftrightarrow D_7$ |
| XV | $(1, 1, 2, 3)[7]$ | $(0, 1, 2, 3)_{e8} + (1, 0, 0, 0)_a$ | $31 = \mathbf{16}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{8}_a$ | $E_8^{(1)} \leftrightarrow A_6^{(1)}$ |
| XVI | $(1, 1, 2, 4)[8]$ | $(0, 1, 2, 3)_{e8} + (1, 0, 0, 1)_d$ | $35 = \mathbf{16}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{12}_d$ | $E_8^{(1)} \leftrightarrow D_8$ |
| $XVII^*$ | $(1, 1, 3, 4)[9]$ | $(0, 1, 2, 3)_{e8} + (1, 0, 1, 1)_{e6}$ | $33 = \mathbf{16}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{10}_{e6}$ | $E_8^{(1)} \leftrightarrow E_6^{(1)}$ |
| $XVIII^*$ | $(1, 1, 3, 5)[10]$ | $(0, 1, 2, 3)_{e8} + (1, 0, 1, 2)_{e7}$ | $36 = \mathbf{16}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{13}_{e7}$ | $E_8^{(1)} \leftrightarrow E_7^{(1)}$ |
| XIX^* | $(1, 1, 4, 6)[12]$ | $(0, 1, 2, 3)_{e8} + (1, 0, 2, 3)_{e8}$ | $39 = \mathbf{16}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{16}_{e8}$ | $E_8^{(1)} \leftrightarrow E_8^{(1)}$ |
| XX^* | $(1, 1, 1, 3)[6]$ | $(0, 2, 1, 3)_{e8} + (2, 0, 1, 3)_{e8}$ | $21 = \mathbf{7}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{7}_{e8}$ | $F_4^{(1)} \leftrightarrow F_4^{(1)}$ |
| XXI^* | $(3, 3, 2, 4)[12]$ | $(0, 3, 1, 2)_{e8} + (3, 0, 1, 2)_{e8}$ | $15 = \mathbf{4}_{e8} + (\sigma = \{7\}_\Delta) + \mathbf{4}_{e8}$ | $G_2^{(1)} \leftrightarrow G_2^{(1)}$ |

mirror symmetry or via a triple expansion of the monomial $P_0 = (6, 6, 6, 0, 0)$ of the Weierstrass triangle, which is the normal intersection of the following three extended vectors from $\mathcal{A}_5^{(3)}$: $\vec{k}_5^{ex} = (0, 0, 1, 2, 3), (0, 1, 0, 2, 3), (1, 0, 0, 2, 3)$. More precisely, the listing of this chain can be obtained from the following Diophantine expansion [35] :

$$D_5[1, 1, 1/3] : P_0 \rightarrow \{P_1, P_2, P_3 : \frac{1}{3}(P_1 + P_2 + P_3) = P_0\}. \quad (108)$$

In this definition, the subscript 5 labels the dimension of the monomial P_0 , and the combination $[111/3]$ labels the type of Diophantine expansion: in this case, the monomial P_0 is decomposed into three monomials with the condition $(P_1 + P_2 + P_3)/3 = P_0$. In Table 2, one can sometimes see that the same vector can give different Coxeter-Dynkin diagrams, for example, 14 and 14'.

Table 2: *The eldest vectors for all S_3 -symmetric chains of CY_3 hypersurfaces.*

| N | $\vec{k}_i(\text{eldest})$ | $h^{2,1}$ | $h^{1,1}$ | V | V^* | $Cox - Dyn$ | $P_0\text{section}$ |
|-----|----------------------------|-----------|-----------|-----|-------|-------------------|---------------------|
| 1 | (1, 1, 1 1, 1) | 101 | 1 | 5 | 5 | $(E_6)(A)^2$ | (3, 1, 1 0, 0) |
| 2 | (1, 1, 1 1, 2) | 103 | 1 | 5 | 5 | $(E_7)(A)^2$ | (4, 1, 1 0, 0) |
| 2' | (1, 1, 1 1, 2) | 103 | 1 | 5 | 5 | $(E_6)(D)(A)$ | (3, 2, 1 0, 0) |
| 3 | (1, 1, 1 1, 3) | 122 | 2 | 8 | 6 | $(E_6)(D)^2$ | (3, 2, 2 0, 0) |
| 4 | (1, 1, 1 1, 4) | 149 | 1 | 5 | 5 | $(E_7)(D)^2$ | (4, 2, 2 0, 0) |
| 5 | (1, 1, 1 2, 2) | 95 | 2 | 9 | 6 | $(E_6)^2(A)$ | (3, 3, 1 0, 0) |
| 6 | (1, 1, 1 2, 3) | 106 | 2 | 8 | 6 | $(E_6)^2(D)$ | (3, 3, 2 0, 0) |
| 6' | (1, 1, 1 2, 3) | 106 | 2 | 8 | 6 | $(E_7)(E_6)(A)$ | (4, 3, 1 0, 0) |
| 6'' | (1, 1, 1 2, 3) | 106 | 2 | 8 | 6 | $(E_8)(A)^2$ | (6, 1, 1 0, 0) |
| 7 | (1, 1, 1 2, 4) | 123 | 3 | 9 | 6 | $(E_7)(E_6)(D)$ | (4, 3, 2 0, 0) |
| 7' | (1, 1, 1 2, 4) | 123 | 3 | 9 | 6 | $(E_8)(D)(A)$ | (6, 2, 1 0, 0) |
| 8 | (1, 1, 1 2, 5) | 145 | 1 | 5 | 5 | $(E_7)^2(D)$ | (4, 4, 2 0, 0) |
| 8' | (1, 1, 1 2, 5) | 145 | 1 | 5 | 5 | $(E_8)(D)^2$ | (6, 2, 2 0, 0) |
| 9 | (1, 1, 1 3, 3) | 112 | 4 | 5 | 5 | $(E_6)^3$ | (3, 3, 3 0, 0) |
| 10 | (1, 1, 1 3, 4) | 126 | 4 | 10 | 7 | $(E_7)(E_6)^2$ | (4, 3, 3 0, 0) |
| 10' | (1, 1, 1 3, 4) | 126 | 4 | 10 | 7 | $(E_8)(E_6)(A)$ | (6, 3, 1 0, 0) |
| 11 | (1, 1, 1 3, 5) | 144 | 4 | 10 | 7 | $(E_7)^2(E_6)$ | (4, 4, 3 0, 0) |
| 11' | (1, 1, 1 3, 5) | 144 | 4 | 10 | 7 | $(E_8)(E_7)(A)$ | (6, 4, 1 0, 0) |
| 12 | (1, 1, 1 3, 6) | 165 | 3 | 5 | 5 | $(E_7)^3$ | (4, 4, 4 0, 0) |
| 12' | (1, 1, 1 3, 6) | 165 | 3 | 5 | 5 | $(E_8)(E_7)(D)$ | (6, 4, 2 0, 0) |
| 13 | (1, 1, 1 4, 5) | 154 | 4 | 7 | 6 | $(E_8)(E_6)^2$ | (6, 3, 3 0, 0) |
| 14 | (1, 1, 1 4, 6) | 173 | 5 | 9 | 6 | $(E_8)^2(A)$ | (6, 6, 1 0, 0) |
| 14' | (1, 1, 1 4, 6) | 173 | 5 | 9 | 6 | $(E_8)(E_7)(E_6)$ | (6, 4, 3 0, 0) |
| 15 | (1, 1, 1 4, 7) | 195 | 3 | 7 | 6 | $(E_8)(E_7)^2$ | (6, 4, 4 0, 0) |
| 15' | (1, 1, 1 4, 7) | 195 | 3 | 7 | 6 | $(E_8)^2(D)$ | (6, 6, 2 0, 0) |
| 16 | (1, 1, 1 5, 7) | 208 | 4 | 7 | 6 | $(E_8)^2(E_6)$ | (6, 6, 3 0, 0) |
| 17 | (1, 1, 1 5, 8) | 231 | 3 | 7 | 6 | $(E_8)^2(E_7)$ | (6, 6, 4 0, 0) |
| 18 | (1, 1, 1 6, 9) | 272 | 2 | 5 | 5 | $(E_8)^3$ | (6, 6, 6 0, 0) |

3.2.3 New Results for CY_4 Spaces

We now apply the technique explained in the earlier subsections to find the arity-4 chains for the 6-dimensional CY_4 case:

$$\omega_4 : \mathbb{A}_6^{(3)} \star \mathbb{A}_6^{(3)} \star \mathbb{A}_6^{(3)} \star \mathbb{A}_6^{(3)} \mapsto A_6^3, \quad (109)$$

where the set \mathbb{A}_6^3 contains vectors constructed from the following five RWVs: $\vec{k}_1 = (1)$, $\vec{k}_2 = (1, 1)$, $\vec{k}_3 = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$ by extending them to dimension 6. It is trivial to see that there are 221 such extended vectors:

$$\begin{aligned}
\vec{k}_1^{(ex)} &= (0, \dots, 0, 1) & + & \quad 6 \text{ permutations}; \\
\vec{k}_2^{(ex)} &= (0, \dots, 1, 1) & + & \quad 15 \text{ permutations}; \\
\vec{k}_3^{(ex)} &= (0, \dots, 1, 1, 1) & + & \quad 15 \text{ permutations}, \\
\vec{k}_3^{(ex)} &= (0, \dots, 1, 1, 2) & + & \quad 60 \text{ permutations}, \\
\vec{k}_3^{(ex)} &= (0, \dots, 1, 2, 3) & + & \quad 120 \text{ permutations},
\end{aligned} \quad (110)$$

Among these 221 extended vectors one should look for the quadruple ‘good’ intersections, which involves considering a total of C_{221}^4 combinatorial possibilities.

Some of the quadruples turn out to be equivalent to other quadruples, after some permutation of the vectors and/or their components. Checking this is by far the most time-consuming task for our computer program. To give some idea, there are 720,768 quadruples that do not all have zeroes in the same component (which is definitely not a good case) and where the number of intersection monomials is between 4 and 10 (the intersection of four 6-dimensional vectors is a polygon, and all good cases on the plane are known to have between 4 and 10 monomials). When checking if two quadruples are equivalent, one has to check $720,768 \times 720,767$ divided by 2 combinations, and this, in general, for all possible permutations of the 4 vectors and the 6 components, *i.e.*, $4! \times 6!$ times. The final result is that, in dimension 6, there are 5,607 different 4-vector chains obtainable by extending the 5 ‘good’ vectors existing in dimensions 1, 2 and 3. Many of these 5,607 chains of arity 4 could be reducible, so the next step is to identify the irreducible chains. This is done later using the method of invariant monomials (IMs).

In this case, the number of irreducible chains turns out to be 2,111, and the number of different eldest RWVs that generate all these 2,111 chains is only 397. This reveals that these 397 eldest vectors have a very rich composite substructure. We show in Table 3 the list of RWVs with maximal symmetry at least as large as S_4 . This Table also shows the structures of the Coxeter-Dynkin diagrams included in the reflexive polyhedra. It is easy to convince oneself that the number of Coxeter-Dynkin factors grows with the arity. In particular, for arity 4, the number of Coxeter-Dynkin diagrams is also equal to 4, as seen in Table 3.

Along the E_r line, the structures of the CY_d surfaces are described by cubic and quartic monomials, and sometimes pairs of conics, as we illustrated in the cases of $K3$ and CY_3 , and show more concretely in the next Section. There we use Diophantine relations between the cubic and quartic IMs to obtain complete information about the chains and eldest vectors (see Fig. 6). The method of IMs is independent of Batyrev’s reflexive polyhedra, and has an exclusively algebraic character.

4 The Monomial Decomposition Route to the Universal Calabi-Yau Algebra

We have seen that the Universal Calabi-Yau Algebra (UCYA) systematizes all the CY_d manifolds according to the arity of their construction as well as their dimensionality. In this way, the UCYA enables us to explore systematically relations between different CY_d spaces,

Table 3: *The eldest vectors for all the S_4 -symmetric arity-4 chains of CY_3 hypersurfaces.*

| N | $\vec{k}_i(\text{eld})$ | Coxeter – Dynkin | P_0 section | N | $\vec{k}_i(\text{eld})$ | Coxeter – Dynkin | section P_0 |
|------|-------------------------|--------------------|---------------|-------|-------------------------|----------------------|---------------|
| 1 | (1, 1, 1, 1, 1, 1) | $(E_6)(A)^3$ | (311100) | 16 | (1, 1, 1, 1, 4, 4) | $(E_6)^4$ | (333300) |
| 2 | (1, 1, 1, 1, 1, 2) | $(E_6)(D)(A)^2$ | (321100) | 17 | (1, 1, 1, 1, 4, 5) | $(E_8)(E_6)^2(A)$ | (633100) |
| 2' | (1, 1, 1, 1, 1, 2) | $(E_7)(A)^3$ | (411100) | 18 | (1, 1, 1, 1, 4, 6) | $(E_7)^2(E_6)^2$ | (443300) |
| 3 | (1, 1, 1, 1, 1, 3) | $(E_6)(D)^2(A)$ | (322100) | | | | |
| 3' | (1, 1, 1, 1, 1, 3) | $(E_7)(D)(A)^2$ | (421100) | 18' | (1, 1, 1, 1, 4, 6) | $(E_8)(E_7)(E_6)(A)$ | (643100) |
| 4 | (1, 1, 1, 1, 1, 4) | $(E_6)(D)^3$ | (322200) | 18'' | (1, 1, 1, 1, 4, 6) | $(E_8)(E_6)^2(D)$ | (633200) |
| 4' | (1, 1, 1, 1, 1, 4) | $(E_7)(D)^2(A)$ | (422100) | 18''' | (1, 1, 1, 1, 4, 6) | $(E_8)^2(A)^2$ | (661100) |
| 5 | (1, 1, 1, 1, 1, 5) | $(E_7)(D)^3$ | (422200) | 19 | (1, 1, 1, 1, 4, 7) | $(E_7)^3(E_6)$ | (444300) |
| 6 | (1, 1, 1, 1, 2, 2) | $(E_6)^2(A)^2$ | (331100) | 19' | (1, 1, 1, 1, 4, 7) | $(E_8)(E_7)^2(A)$ | (644100) |
| 7 | (1, 1, 1, 1, 2, 3) | $(E_6)^2(D)(A)$ | (332100) | 19'' | (1, 1, 1, 1, 4, 7) | $(E_8)^2(D)(A)$ | (662100) |
| 7' | (1, 1, 1, 1, 2, 3) | $(E_7)(E_6)(A)^2$ | (431100) | 20 | (1, 1, 1, 1, 4, 8) | $(E_7)^4$ | (444400) |
| 7'' | (1, 1, 1, 1, 2, 3) | $(E_8)^2(A)^3$ | (611100) | 20' | (1, 1, 1, 1, 4, 8) | $(E_8)(E_7)^2(D)$ | (644200) |
| 8 | (1, 1, 1, 1, 2, 4) | $(E_6)^2(D)^2$ | (332200) | 20'' | (1, 1, 1, 1, 4, 8) | $(E_8)^2(D)^2$ | (662200) |
| 8' | (1, 1, 1, 1, 2, 4) | $(E_7)^2(A)^2$ | (441100) | 21 | (1, 1, 1, 1, 5, 6) | $(E_8)(E_6)^3$ | (633300) |
| 8'' | (1, 1, 1, 1, 2, 4) | $(E_8)(D)(A)^2$ | (621100) | 22 | (1, 1, 1, 1, 5, 7) | $(E_8)(E_7)(E_6)^2$ | (643300) |
| 9 | (1, 1, 1, 1, 2, 5) | $(E_7)^2(D)(A)$ | (442100) | 22' | (1, 1, 1, 1, 5, 7) | $(E_8)(E_7)^2(D)$ | (644200) |
| 9' | (1, 1, 1, 1, 2, 5) | $(E_7)(E_6)(D)^2$ | (432200) | 22'' | (1, 1, 1, 1, 5, 7) | $(E_8)^2(E_6)(A)$ | (663100) |
| 9'' | (1, 1, 1, 1, 2, 5) | $(E_8)(D)^2(A)$ | (622100) | 23 | (1, 1, 1, 1, 5, 8) | $(E_8)(E_7)^2(E_6)$ | (644300) |
| 10 | (1, 1, 1, 1, 2, 6) | $(E_7)^2(D)^2$ | (442200) | 23' | (1, 1, 1, 1, 5, 8) | $(E_8)^2(E_7)(A)$ | (664100) |
| 10' | (1, 1, 1, 1, 2, 6) | $(E_8)(D)^3$ | (622200) | 23'' | (1, 1, 1, 1, 5, 8) | $(E_8)^2(E_6)(D)$ | (663200) |
| 11 | (1, 1, 1, 1, 3, 3) | $(E_6)^3(A)$ | (333100) | 24 | (1, 1, 1, 1, 5, 9) | $(E_8)(E_7)^3$ | (644400) |
| 12 | (1, 1, 1, 1, 3, 4) | $(E_6)^3(D)$ | (333200) | 25 | (1, 1, 1, 1, 6, 8) | $(E_8)^2(E_6)^2$ | (663300) |
| 12' | (1, 1, 1, 1, 3, 4) | $(E_7)(E_6)^2(A)$ | (433100) | 26 | (1, 1, 1, 1, 6, 9) | $(E_8)^2(E_7)(E_6)$ | (664300) |
| 12'' | (1, 1, 1, 1, 3, 4) | $(E_8)(E_6)(A)^2$ | (631100) | 26' | (1, 1, 1, 1, 6, 9) | $(E_8)^3(A)$ | (666100) |
| 13 | (1, 1, 1, 1, 3, 5) | $(E_7)^2(E_6)(A)$ | (443100) | 27 | (1, 1, 1, 1, 6, 10) | $(E_8)^3(D)$ | (666200) |
| 13' | (1, 1, 1, 1, 3, 5) | $(E_8)(E_6)(D)(A)$ | (632100) | 27' | (1, 1, 1, 1, 6, 10) | $(E_8)^2(E_7)^2$ | (664400) |
| 13'' | (1, 1, 1, 1, 3, 5) | $(E_8)(E_7)(A)^2$ | (641100) | 28 | (1, 1, 1, 1, 7, 10) | $(E_8)^3(E_6)$ | (666300) |
| 14 | (1, 1, 1, 1, 3, 6) | $(E_7)^3(A)$ | (444100) | 29 | (1, 1, 1, 1, 7, 11) | $(E_8)^3(E_7)$ | (666400) |
| 14' | (1, 1, 1, 1, 3, 6) | $(E_8)(E_7)(D)(A)$ | (642100) | 30 | (1, 1, 1, 1, 8, 12) | $(E_8)^4$ | (666600) |
| 15 | (1, 1, 1, 1, 3, 7) | $(E_7)^3(D)$ | (444200) | | | | |
| 15' | (1, 1, 1, 1, 3, 7) | $(E_8)(E_7)(D)^2$ | (642200) | | | | |

as well as their internal structures, such as the singularities which are related to possible new gauge-group structures. In the previous section, we concentrated on the normal expansion technique, related to Batyrev's reflexive polyhedron method, that inter-relates CY_d spaces in different numbers of dimensions. We also gave some examples of the extra information that can be gleaned by decomposing IMs. In this Section, we explore further this technique based on the Diophantine expansion of invariant monomials (IMs), illustrated in Fig. 6, that relates spaces with different arities. As we show later in more detail, this algebraic method uses the Diophantine properties of the equations defining Calabi-Yau spaces. It enables us to construct and enumerate the chains of CY_d spaces that we discussed previously. Moreover, we find recurrence relations for the numbers of chains with different arities in different dimensions.

4.1 Analysis of the A_r and D_r Lines

We start with some simple examples, taken from the leading diagonal A_r line, corresponding to the maximal arity $r = n$. As we have already discussed, the IMs along this line have a very simple structure, since in each dimension n there is just one such monomial, the unit $E_n = (1, \dots, 1)_n$: see Table 4. To find the chain of CY_d spaces corresponding to each such IM, we should solve the equation

$$\vec{k}^{i(ex)} \cdot E_n = d(\vec{k}^{i(ex)}) \quad (111)$$

which immediately gives us

$$(\vec{k}^{i(ex)})_1 + \dots + (\vec{k}^{i(ex)})_n = d(\vec{k}^{i(ex)}), \quad (112)$$

where $(\vec{k}^{i(ex)})_1, \dots, (\vec{k}^{i(ex)})_n$ are the n components of the extended vector $\vec{k}^{i(ex)}$. In general, such an equation has an infinite number of solutions, but these are restricted to a finite number by the fact that in the UCYA the extended vectors along this diagonal line can be constructed only by extending the unit vector $\vec{k}_1 = (1)$ to dimension n . In this simple case the Diophantine equation can immediately be rewritten in the following form:

$$(\vec{k}^{i(ex)})_1 + \dots + (\vec{k}^{i(ex)})_n = 1, \quad (113)$$

which gives exactly n different extended vectors: $(\vec{k}^{i(ex)})_j = \delta_j^i$, where $i, j = 1, 2, \dots, n$. These n different solutions $\vec{k}^{i(ex)}$, ($i = 1, \dots, n$), actually produce only one chain of arity n , with the eldest vector

$$\vec{k}_n = \cup_{i=1}^{i=n} (0, \dots, 0, 1_i, 0, \dots) = (1, \dots, 1). \quad (114)$$

One can check this result by the previous method of looking for the intersection of these n extended vectors:

$$\cap_{i=1}^{i=n} (0, \dots, 0, 1_i, 0, \dots) = (1')_n. \quad (115)$$

Thus, one can see that the unit IM in any dimension n uniquely determines a single chain of arity n with unit reflexive eldest vector.

It is a consequence of the UCYA that all RWVs can be decomposed in terms of the extended vectors of this A_r line:

$$\vec{k}_n = \sum_{i=1}^{i=n} a_i \vec{k}_1^{i(ex)}, \quad (116)$$

with some non-negative coefficients a_i . The problem of finding the coefficients in this decomposition is connected directly with the internal structure of the CY_d spaces. For instance, if

the CY_d space had only a simple substructure related just to the unit invariant monomial E_n ³, one might think that the complete set of CY_d spaces could be found by simple extension of this unit E_n monomial to n monomials P_i , with the property $(1/n)(P_1 + \dots + P_n) = E_n$, in which case the genome for all CY_d spaces would be solved. However, fortunately CY_d spaces have a much richer and more intriguing internal substructure, and such a direct Diophantine expansion cannot give the full number and substructure of all the CY_d spaces. The Diophantine expansions of the unit monomials E_4 in the $K3$ case and E_5 in the CY_3 case give only the 42 and 7,269 RWVs in these two chains, respectively.

Due to the complicated internal structure of CY_d spaces, the Diophantine expansions can in many cases be degenerate. In these cases, the E_n cannot simply be expanded in terms of n points $P_i : i = 1, \dots, n$, with the property $E_n = (1/n)(P_1 + \dots + P_n)$. However, some other weaker conditions could still apply:

$$E_n \mapsto \{P_1, \dots, P_{(n-r+1)} | E_n = \frac{1}{(n-r+1)}(P_1 + \dots + P_{n-r+1})\} \quad (117)$$

for $r = (n-1), (n-2), \dots, 3, 2$. This is another way to see why we must go on to study further the substructure of the CY_d spaces, corresponding to the arities $r = (n-1), (n-2), \dots, 3, 2$, i.e., we should go on to study points on the next lines on the arity-dimension plot. The corresponding CY_d spaces will have substructures described by higher-degree monomials, such as conics, cubics, quartics, quintics, etc.. For example, the CY_d spaces with substructures corresponding to the E_r line have elliptic fibres, whose substructure is described by cubic and quartic monomials. The CY_d spaces with richer substructure, corresponding to the next (fourth) line, have $K3$ fibres, and are described by the properties of quintic, sextic, septic and octic monomials. Some of these CY_d spaces also have a hyperelliptic substructure.

These arguments show that, in order to study the structures of the chains of fixed arity and dimension appearing along the next lines: D_r, E_r, \dots , we should first determine the types of IMs corresponding to these lines. Next, we should find suitable sets of extended vectors by solving the following equations:

$$\vec{k}^{i(ex)} \cdot (IM)_a = d(\vec{k}^{i(ex)}). \quad (118)$$

In looking for the solutions of these equations, one may use the fact that, in the UCYA, each sloping line is described by a particular type of extended weight vectors.

We consider the CY_d spaces with substructures corresponding to the second D_r line with arity $r = (n-1)$. We have the following sets of possible different types of conic monomials in

³The single central point in one of the Batyrev polyhedra.

any dimension n , as given in the Table 4, which can be generalized by a recurrence relation to any dimension n :

$$N_{conics} = \frac{(n)(n-1)}{2}. \quad (119)$$

We propose to determine all the possible conic IMs for this line, starting from the unit monomial E_n and two conic monomials, $C_{i(n)}$ and $C_{j(n)}$, taken from Table 4 ⁴. These monomials should satisfy the following Diophantine property: $r = (n-1)$:

$$C_{i(n)} + C_{j(n)} = \frac{1}{2}E_n, \quad (120)$$

where the index n notes the dimension being considered.

Table 4: *The invariant monomials (linear and quadratic) for the (1)+(11) RWVs extended to Weierstrass, K3, CY₃ and CY₄ spaces, corresponding to the D_r line on the arity-dimension plot 3. In the case of the A_r line, there can be only linear invariant monomials.*

| P | L | W | K3 | K3 | Qu | in | tic | Se | x | t | ic |
|-----|------|-------|--------|------------------|---------|--------------------|-------------------------------|----------|----------------------|----------------------------------|--|
| (1) | (11) | (111) | (1111) | | (11111) | | | (111111) | | | |
| | (20) | (200) | (2000) | | (20000) | | | (200000) | | | |
| | | (210) | (2100) | (2110) | (21000) | (21100) | (21110) | (210000) | (211000) | (211100) | (211110) |
| | | (220) | (2200) | (2210) (2220) | (22000) | (22100) (22200) | (22110) (22210) (22220) | (220000) | (221000) (222000) | (221100) (222100) (222200) | (221110) (222110) (222210) (222220) |

In order to enumerate these chains, we should find all possible pairs of conic monomials with the required Diophantine property, solving the following equations:

$$\vec{k}^{i(ex)} \cdot C_{1(n)} = \vec{k}^{i(ex)} \cdot E_n = d(\vec{k}^{i(ex)}). \quad (121)$$

To give sense to these equations and, consequently, to evaluate the finite number of solutions for the chains and their eldest vectors corresponding to the arity $r = (n-1)$, one should recall that, in the UCYA, the points on this line are determined by n -dimensional extensions of the two eldest vectors $\vec{k}_1 = (1)$ and $\vec{k}_2 = (1, 1)$. This means that the possible values of $d(\vec{k}^{i(ex)})$ in these equations are only 1 and 2. Also, the components of the extended vectors can only be 0, 1 or 2. The n -dimensional recurrence relation for conic monomials allows one to find all the IMs and to solve this problem for all dimensions n , in the sense of finding all the corresponding eldest vectors. The recurrence formula for the numbers of chains along this line is:

$$\begin{aligned} N_{chains} &= k \cdot (k+1), & \text{if } n &= (2k+1) \\ N_{chains} &= k^2, & \text{if } n &= (2k). \end{aligned} \quad (122)$$

⁴In this Table, one can use all possible permutations of the expressions for the conic monomials.

Accordingly, along this line the numbers of the eldest vectors and chains in the dimensions $n = 2, 3, 4, \dots$ are the following: 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100, 110, 121, 132, 144,

In order to understand this recurrence formula, one can construct the list of all different pairs of conics that appear in any dimension, as illustrated in Table 4. For example, for CY_3 ($n = 5$) one can get $6 = 2 \cdot 3$ chains of arity 4, for CY_4 ($n = 6 = 2k$) there are $9 = 3^2$ chains of arity 5 and for CY_5 ($n = 7 = 2k + 1, k = 3$) there are $12 = 3 \cdot 4$ chains of arity 6. Thus, one of the main results of the IM approach for the first two diagonal lines, A_r and D_r , is to solve this aspect of the CY_d ‘genome’ in any dimension.

There is another interesting observation connected to the one-to-one correspondence between the IMs and chains. As an example, consider the following pair of n -dimensional conics:

$$C_{1(n)} = (2, 2, 2, \dots, 2, 0), \quad C_{2(n)} = (0, 0, 0, \dots, 0, 2). \quad (123)$$

The solutions of the equations

$$\vec{k}^{i(ex)} \cdot C_{1(n)} = \vec{k}^{i(ex)} \cdot E_n = d(\vec{k}^{i(ex)}). \quad (124)$$

can be expressed in the following form:

$$(\vec{k}^{i(ex)})_1 + \dots + (\vec{k}^{i(ex)})_{n-1} = 1, \quad (\vec{k}^{i(ex)})_n = 1. \quad (125)$$

So we get exactly $(n - 1)$ different solutions for the extended vectors, producing one chain of arity $r = (n - 1)$:

$$\begin{aligned} \vec{k}^{1(ex)} &= (1, 0, 0, \dots, 0, 0, 1) \\ \vec{k}^{2(ex)} &= (0, 1, 0, \dots, 0, 0, 1) \\ &\dots = \dots \\ \vec{k}^{n-2(ex)} &= (0, 0, 0, \dots, 1, 0, 1) \\ \vec{k}^{n-1(ex)} &= (0, 0, 0, \dots, 0, 1, 1) \end{aligned} \quad (126)$$

with the eldest vector $\vec{k}_n = (1, 1, 1, \dots, 1, 1, n - 1)$.

To find the sets of RWVs inside the chains, one may study the following Diophantine condition for the conics:

$$C_{1(n)} \mapsto \{P_1, \dots, P_{(n-1)} | C_{1(n)} = \frac{1}{(n-1)}(P_1 + \dots P_{(n-1)})\} \quad (127)$$

and then solve the following n equations for the eldest vectors:

$$\begin{aligned}\vec{k}^{i(ex)} \cdot P_j &= \vec{k}^{i(ex)} \cdot C_{2(n)} = d(\vec{k}^{i(ex)}), \\ C_{1(n)} + C_{2(n)} &= \frac{1}{2}E_n.\end{aligned}\tag{128}$$

In the case of $K3$ chains, we can compare the list of the RWVs obtained from the Diophantine expansion with the complete list, which contains 34 and 48 possibilities, respectively. We can see that, in this case, there are 14 RWVs that cannot be obtained directly by the expansion of conics $C_{1(n)}$ in terms of order- $(n-1)$ monomials. Therefore, we should study the possibilities for expanding in terms of order- $(n-2), (n-3), \dots$ monomials, going to the next diagonal lines of the arity-dimension plot. Due to the more complicated structure of $CY_{d=(n-2)}$ spaces with other arities $r = (n-2), \dots, 3, 2$, corresponding to the third E_r line, the fourth line, etc., the Diophantine condition for the conics is not closed.

4.2 Analysis of the E_r Line.

As the next step, we consider the situation along the diagonal E_r line.

4.2.1 Recurrences of Sets of Invariant Monomials

Continuing our previous approach, the first step is to enumerate the cubic and quartic monomials, from which we can find all the IMs along this E_r line. The appearance of cubic monomials is connected with the following new Diophantine condition for the expansion of the unit monomials E_n of the A_r line:

$$E_n \mapsto \{P_1, P_2, P_3 | \frac{1}{3}(P_1 + P_2 + P_3) = E_n\}.\tag{129}$$

However, the set of appropriate cubic monomials is somewhat more restricted. Similarly, the appearance of quartic monomials is connected with the possible Diophantine expansion of the conic monomials $Ci(n)$ of the second D_r line:

$$C_{i(n)} \mapsto \{P_1, P_2 | \frac{1}{2}(P_1 + P_2) = C_{i(n)}\}.\tag{130}$$

Again, there are some further restrictions on the list of possible quartic monomials. We now give the list of cubic and quartic monomials that are relevant in the framework of the UCYA.

Examining Tables 5 and 6, one can convince oneself that there exist recurrence formulae for the IMs in any dimension. These formulae are obvious for the leading (red and green)

Table 5: *The invariant monomials (cubics and quartics) for the five weight vectors (1)+(11)+(111)+(112)+(123), extended to Weierstrass, K3, CY₃ and CY₄ spaces, corresponding to the E_r line on the arity-dimension plot.*

| <i>W</i> | <i>K3</i> | <i>K3</i> | <i>N</i> | <i>Quin</i> | <i>ti</i> | <i>c</i> | <i>N</i> | <i>Se</i> | <i>xt</i> | <i>i</i> | <i>c</i> | <i>N</i> |
|----------|-----------|-----------|----------|-------------|-----------|----------|----------|-----------|-----------|----------|----------|----------|
| (300) | (3000) | | 1 | (30000) | | | 1 | (300000) | | | | 1 |
| (310) | (3100) | (3110) | 2 | (31000) | (31100) | (31110) | 3 | (310000) | (311000) | (311100) | (311110) | 4 |
| | (3200) | (3210) | | (32000) | (32100) | (32110) | | (320000) | (321000) | (321100) | (321110) | |
| | | | 2 | | (32200) | (32210) | 5 | | (322000) | (322100) | (322110) | 9 |
| | | | | | | | | | | (322200) | (322210) | |
| | (3300) | (3310) | | (33000) | (33100) | (33110) | | (330000) | (331000) | (331100) | (331110) | |
| | | | | | (33200) | (33210) | | | (332000) | (332100) | (332110) | |
| | | | 2 | | | (33300) | 7 | | (333000) | (333100) | (333110) | |
| | | | | | | | | | | (333200) | (333210) | 16 |
| | | | | | | | | | | (333300) | (333310) | |
| (400) | (4000) | | 1 | (40000) | | | 1 | (400000) | | | | 1 |
| | (4100) | (4110) | 2 | (41000) | (41100) | (41110) | 3 | (410000) | (411000) | (411100) | (411110) | 4 |
| | (4200) | (4210) | | (42000) | (42100) | (42110) | | (420000) | (421000) | (421100) | (421110) | |
| | | | 2 | | (42200) | (42210) | 5 | | (422000) | (422100) | (422110) | 9 |
| | | | | | | | | | | (422200) | (422210) | |
| | (4300) | (4310) | | (43000) | (43100) | (43110) | | (430000) | (431000) | (431100) | (431110) | |
| | | | | | (43200) | (43210) | | | (432000) | (432100) | (432110) | |
| | | | 2 | | | (43300) | 7 | | (433000) | (433100) | (433110) | |
| | | | | | | | | | | (433200) | (433210) | 16 |
| | | | | | | | | | | (433300) | (433310) | |
| | (4400) | (4410) | | (44000) | (44100) | (44110) | | (440000) | (441000) | (441100) | (441110) | |
| | | | | | (44200) | (44210) | | | (442000) | (442100) | (442110) | |
| | | | 2 | | | (44300) | 9 | | (443000) | (443100) | (443110) | |
| | | | | | | (44400) | | | (444000) | (444100) | (444110) | |
| | | | | | | | | | | (444200) | (444210) | |
| | | | | | | | | | | (444300) | (444310) | |
| | | | | | | | | | | (444400) | (444410) | 25 |

lines. The expressions for the numbers of cubic and quartic monomials are, respectively:

$$\begin{aligned}
N_{cubics} &= \frac{1}{6}(n-2)(n-1)(n+3) \\
N_{quartics} &= \frac{1}{24}(n-2)(n-1)(n)(n+5)
\end{aligned} \tag{131}$$

There are remarkable links between the numbers of conics, cubics and quartics. For example, to obtain the number of quartics in dimension n , one should sum over all the cubics in dimensions $3, 4, \dots, n$, i.e., $N_{Quart}^{(n)} = \sum_i 3^{i=n} N_{Cub}^{(i)}$. Thus, as seen in Fig. 7, the number 105 of quartic monomials in the septic Calabi-Yau case can be represented as follows: $2_{dim=3} + 7_{dim=4} + 16_{dim=5} + 30_{dim=6} + 50_{dim=7}$.

Based on these tables, one can convince oneself that there should also exist n -dimensional recurrence formulae for the IMs applicable along other diagonal lines in any dimension, as

we have found for the first two lines on the arity-dimension plot in Fig. 3. However, the situation can become complicated, because, in the construction of the cubic and quartic IMs, one must also take into account conic and conic + cubic monomials, respectively. In the case of Calabi-Yau spaces with Weierstrass fibres, it is also important to know the list of sextic monomials that we give in Table 6. The recurrence formula for the number of sextic monomials can be obtained from this Table:

$$C_{n+2}^{m-3} = \frac{(n+2)!}{(n-3)!5!}, \quad (132)$$

where $n \geq 3$ is the dimension of the weight-vector space.

Table 6: *The sextic monomials, M_6 , for the five weight vectors $(1) + (11) + (111) + (112) + (123)$, extended to $K3$, CY_3 and CY_4 spaces, corresponding to the E_r line on the arity-dimension plot.*

| $K3$ | N | $Quin$ | tic | N | Se | xt | ic | N |
|--------|-----|---------|---|-----|----------|--|--|-----|
| (6000) | 1 | (60000) | | 1 | (600000) | | | 1 |
| (6100) | 1 | (61000) | (61100) | 2 | (610000) | (611000) | (611100) | 3 |
| (6200) | 1 | (62000) | (62100) (62200) | 3 | (620000) | (621000) (622000) | (621100) (622100) (622200) | 6 |
| (6300) | 1 | (63000) | (63100) (63200) (63300) | 4 | (630000) | (631000) (632000) (633000) | (631100) (632100) (632200) (633100) (633200) (633300) | 10 |
| (6400) | 1 | (64000) | (64100) (64200) (64300) (64400) | 5 | (640000) | (641000) (642000) (643000) (644000) | (641100) (642100) (642200) (643100) (643200) (643300) (644100) (644200) (644300) (644400) | 15 |
| (6600) | 1 | (66000) | (66100) (66200) (66300) (66400) (66600) | 6 | (660000) | (661000) (662000) (663000) (664000) (666000) | (661100) (662100) (662200) (663100) (663200) (663300) (664100) (664200) (664300) (664400) (666100) (666200) (666300) (666400) (666600) | 21 |

4.2.2 Applications to $K3$ Spaces and their Fibrations

We now consider the $K3$ case in more detail, studying the third E_r line. From previous work, we already know that there are 22 chains of arity 2, a result we now rederive using the IM method. To obtain all the chains and the corresponding eldest weight vectors, one needs to consider three types of Diophantine expansions, as described above. The first mechanism to consider is the expansion of the unit monomial in terms of three monomials, whose maximal degree is 3:

$$E_4 \longrightarrow \{P_1, P_2, P_3 : \frac{1}{3}(P_1 + P_2 + P_3) = E_4\}. \quad (133)$$

We see easily from Tables 7, 8 and 9 that there exist 28 cubic IMs, including the single unit monomial E_4 and three monomials P_1, P_2, P_3 .

To find these chains, one should first solve the following equations:

$$\vec{k}^{i(ex)} \cdot (IM)_a = d(\vec{k}^{i(ex)}), \quad (134)$$

where $a = 1, 2, 3$. To solve these equations and to obtain 19 of the chains, we should look for extended vector solutions among the set of 4-dimensional extensions of the following RWVs:

$$\vec{k}_1 = (1); \vec{k}_2 = (1, 1); \vec{k}_3 = (1, 1, 1), \vec{k}_3 = (1, 1, 2), \vec{k}_3 = (1, 2, 3). \quad (135)$$

The second way, that completes the structure of $K3$ spaces along this line, is to consider quartic monomials. This can be done in two steps, first expanding the unit monomial in terms of conics and then expanding the conics in terms of quartics:

$$\begin{aligned} E_4 &\longrightarrow \{C_1, C_2 : \frac{1}{2}(C_1 + C_2) = E_4\} \\ C_1 &\longrightarrow \{P_1, P_2 : \frac{1}{2}(P_1 + P_2) = E_4\}. \end{aligned} \quad (136)$$

We refer to this expansion as the Diophantine $D_4[1, 1, 2/4]$ expansion, and call the expansion through cubics the $D_4[1, 1, 1/3]$ expansion.

Twenty chains can be found this way by solving the following equations:

$$\begin{aligned} \vec{k}^{i(ex)} \cdot P_1 &= \vec{k}^{i(ex)} \cdot P_2 = d(\vec{k}^{i(ex)}), \\ \vec{k}^{i(ex)} \cdot C_2 &= d(\vec{k}^{i(ex)}), \end{aligned} \quad (137)$$

where we look again for solutions among the extended vectors. The $D_4[1, 1, 1/3]$ and $D_4[1, 1, 2/4]$ expansions give 20 chains each, which combine to give the full 22 chains. The last type of Diophantine expansion is the double conic expansion, which is described by the expression $D_4[1, 1/2] + D'_4[1, 1/2]$, i.e.:

$$\begin{aligned} E_4 &\longrightarrow \{C_1, C_2 : \frac{1}{2}(C_1 + C_2) = E_4\} \\ E_4 &\longrightarrow \{C_3, C_4 : \frac{1}{2}(C_3 + C_4) = E_4\} \end{aligned} \tag{138}$$

Solving the corresponding equations leads to 14 such cases, which are contained among the 22 already found.

We thus see that the IM method yields the same 22 chains of $K3$ spaces that were found previously via the normal expansion technique: indeed, just two types of Diophantine expansion were sufficient. To understand these expansions more deeply, look at Tables 7, 8, 9. The monomials P_1, P_2, P_3 , corresponding to the $D_4[1, 1, 1/3]$ Diophantine expansion, together with the unit monomial E_4 , produce the full set of third-order IMs for $K3$. The numbers 3 and 4 we term the degrees of the corresponding IMs. The monomials P_1, P_2, C_2 together with the unit monomial E_4 produce the set of fourth-degree IMs, and these Tables show the different types of third- and fourth-degree IMs.

The third-degree IMs can be involved in the following four types of triples of cubic and/or conic monomials: $(Cub)^3, (Cub)^2(Con), (Cub)(Con)^2$ and $(Con)^3$. Similarly, one can characterize according to the choice of P_1 and P_2 the following six types of fourth-degree IMs: $(Qrt)^2, (Qrt)(Cub), (Qrt)(Con), (Cub)^2, (Cub)(Con)$ and $(con)^2$. These tables show a very important correlation between the types of IMs and the type of planar intersection, which is indicated in these Tables by the special index of the number N . Thus, one can see that the chains $III - \{4\}_\Delta$ and $XII - \{5\}_\Delta$ are determined by the third-degree IMs, $(Cub)^3$ and $(Cub)^2(Con)$, respectively. Correspondingly, the chains $VIII - \{5\}_\Delta$ and $IX - \{5\}_\Delta$ are determined by the fourth-degree IMs, $(Qrt)^2$, and differ from each other by the types of conic, C_2 . We note that the 7 Weierstrass chains, $XV - \{7\}_\Delta, \dots, XXI - \{7\}_\Delta$ have the third-degree and fourth-degree IM types $(Cub)^2(Con)$ and $(Qrt)(Cub)$, respectively.

In the next five Tables, we illustrate the relation between the types of intersections and the types of IMs that determine the 22 $K3$ chains. We divided the 22 chains into sets according to the type of arity-2 intersection. Due to mirror $\sigma \leftrightarrow \pi$ symmetry, these intersections indicate immediately the type of fibre structure of the $K3$ space or its dual $K3^*$. In the holomorphic

quotient approach, a fibre is determined by the type of intersection in the mirror partner of the CY_d space [38]. In cases where $\sigma = \pi$, it directly determines the fibre structure of the CY_d space, and in the opposite case it determines the fibre structure of the mirror Calabi-Yau manifold. In the following Tables we mainly follow the notations of our previous article [34]. For example, the notion $\{10\}_\Delta$ means that the intersection of arity-2 corresponds to the 10 point-monomials producing the Batyrev reflexive triangle, the $\{9\}_\square$ corresponds to the 9 point-monomials producing the reflexive planar square figure, etc.. From these Tables, using the equations $\vec{k}^{i(ex)} \cdot IM_a = d_i$, one can easily convince oneself of the equivalence between the Batyrev reflexive polyhedra description and the IM approach. For these cases, the number of IMs is $a = 3$ and the number of solutions is $i = 2$.

Table 7: *The list of K3 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_4$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_4$, $1/2(C_1 + C_2) = E_4$. Here we present the chains with triangle $\{\sigma = 10, 4\}$ and $\{\sigma = 9, 5\}$ intersections [34].*

| N | $\vec{k}_i(eld)$ | Chain | P_1 | P_2 | P_3/C_2 | IM | IM - type |
|-----------------------|--------------------|-------------------------------|----------------|----------------|----------------|---------|----------------|
| $I - \{10\}_\Delta$ | $(1, 1, 1, 1)[4]$ | $(0, 1, 1, 1) + (1, 0, 0, 0)$ | $(1, 3, 0, 0)$ | $(1, 0, 3, 0)$ | $(1, 0, 0, 3)$ | $\{3\}$ | $(Cub)^3$ |
| | | | $(3, 0, 0, 1)$ | $(0, 2, 1, 1)$ | $(0, 1, 2, 1)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(1, 2, 0, 1)$ | $(0, 1, 2, 1)$ | $(2, 0, 1, 1)$ | $\{3\}$ | $(Con)^3$ |
| | | | $(1, 0, 0, 3)$ | $(1, 2, 0, 1)$ | $(1, 1, 2, 0)$ | $\{4\}$ | $(Cub)(Con)$ |
| $II - \{10\}_\Delta$ | $(1, 1, 2, 2)[6]$ | $(0, 1, 1, 1) + (1, 0, 1, 1)$ | $(3, 3, 0, 0)$ | $(0, 0, 3, 0)$ | $(0, 0, 0, 3)$ | $\{3\}$ | $(Cub)^3$ |
| | | | $(3, 3, 0, 0)$ | $(0, 0, 2, 1)$ | $(0, 0, 1, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(0, 0, 0, 3)$ | $(2, 2, 1, 0)$ | $(1, 1, 2, 0)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(2, 2, 1, 0)$ | $(1, 1, 0, 2)$ | $(0, 0, 2, 1)$ | $\{3\}$ | $(Con)^3$ |
| | | | $(0, 0, 0, 3)$ | $(2, 2, 0, 1)$ | $(1, 1, 2, 0)$ | $\{4\}$ | $(Cub)(Con)$ |
| | | | $(3, 3, 0, 0)$ | $(1, 1, 2, 0)$ | $(0, 0, 1, 2)$ | $\{4\}$ | $(Cub)(Con)$ |
| $III - \{4\}_\Delta$ | $(3, 1, 2, 3)[9]$ | $(0, 1, 1, 1) + (3, 0, 1, 2)$ | $(2, 3, 0, 0)$ | $(1, 0, 3, 0)$ | $(0, 0, 0, 3)$ | $\{3\}$ | $(Cub)^3$ |
| $IV - \{9\}_\Delta$ | $(1, 1, 1, 2)[5]$ | $(0, 1, 1, 2) + (1, 0, 0, 0)$ | $(1, 3, 0, 1)$ | $(1, 0, 2, 1)$ | $(1, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(1, 4, 0, 0)$ | $(1, 0, 4, 0)$ | $(1, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | | $(1, 3, 1, 0)$ | $(1, 1, 3, 0)$ | | $\{4\}$ | $(Cub)^2$ |
| | | | $(1, 4, 0, 0)$ | $(1, 0, 0, 2)$ | $(1, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Con)$ |
| $V - \{9\}_\Delta$ | $(1, 1, 1, 3)[6]$ | $(0, 1, 1, 2) + (1, 0, 0, 1)$ | $(2, 3, 1, 0)$ | $(1, 0, 2, 1)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(2, 4, 0, 0)$ | $(2, 0, 4, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | | $(2, 3, 1, 0)$ | $(2, 1, 3, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Cub)^2$ |
| | | | $(2, 4, 0, 0)$ | $(0, 0, 0, 2)$ | $(1, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Con)$ |
| $VI - \{9\}_\Delta$ | $(1, 1, 2, 4)[8]$ | $(0, 1, 1, 2) + (1, 0, 1, 2)$ | $(3, 3, 1, 0)$ | $(0, 0, 2, 1)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(1, 1, 3, 0)$ | $(2, 2, 0, 1)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(4, 4, 0, 0)$ | $(0, 0, 4, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | | $(3, 3, 1, 0)$ | $(1, 1, 3, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Cub)^2$ |
| | | | $(4, 4, 0, 0)$ | $(0, 0, 0, 2)$ | $(0, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Con)$ |
| | | | | | | | |
| $VII - \{9\}_\Delta$ | $(1, 1, 1, 1)[4]$ | $(0, 1, 1, 2) + (2, 1, 1, 0)$ | $(0, 3, 1, 0)$ | $(2, 0, 0, 2)$ | $(1, 0, 2, 1)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(0, 4, 0, 0)$ | $(0, 0, 4, 0)$ | $(2, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | | $(0, 3, 1, 0)$ | $(0, 1, 3, 0)$ | $(2, 0, 0, 2)$ | $\{4\}$ | $(Cub)^2$ |
| | | | $(0, 4, 0, 0)$ | $(2, 0, 0, 2)$ | $(1, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Con)$ |
| $VIII - \{5\}_\Delta$ | $(1, 2, 3, 2)[8]$ | $(0, 1, 1, 2) + (1, 1, 2, 0)$ | $(4, 0, 0, 2)$ | $(0, 4, 0, 0)$ | $(0, 0, 2, 1)$ | $\{4\}$ | $(Qrt)^2$ |
| $IX - \{5\}_\Delta$ | $(2, 2, 1, 5)[10]$ | $(0, 1, 1, 2) + (2, 1, 0, 3)$ | $(3, 0, 4, 0)$ | $(1, 4, 0, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |

We see from these Tables that the IMs determine completely the fibration structures of the 22 K3 chains:

Table 8: *The list of K3 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_4$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_4$, $1/2(C_1 + C_2) = E_4$. Here we present the chains with ‘quadratic-rhombus’ intersections σ [34].*

| N | $\vec{k}_i(\text{eld})$ | chain | P_1 | P_2 | P_3/C_2 | IM | $IM - \text{type}$ |
|------------------------|-------------------------|-------------------------------|----------------|----------------|----------------|-----------|--------------------|
| $X - \{9\}_\square$ | $(1, 1, 1, 1)[4]$ | $(0, 0, 1, 1) + (1, 1, 0, 0)$ | $(2, 0, 2, 0)$ | $(1, 1, 0, 2)$ | $(0, 2, 1, 1)$ | $\{3\}$ | $(Con)^3$ |
| | | | $(2, 0, 2, 0)$ | $(2, 0, 0, 2)$ | $(0, 2, 1, 1)$ | $\{4\}$ | $(Con)^2$ |
| | | | $(2, 0, 2, 0)$ | $(0, 2, 0, 2)$ | | $\{2+2\}$ | $[(Con)^2]^2$ |
| | | | $(2, 0, 0, 2)$ | $(0, 2, 2, 0)$ | | $\{2+2\}$ | $[(Con)^2]^2$ |
| $XI - \{9\}_\square$ | $(1, 1, 1, 2)[5]$ | $(0, 0, 1, 1) + (1, 1, 0, 1)$ | $(3, 0, 2, 0)$ | $(0, 2, 1, 1)$ | $(0, 1, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(2, 1, 2, 0)$ | $(0, 2, 1, 1)$ | $(1, 0, 0, 2)$ | $\{3\}$ | $(Con)^3$ |
| | | | $(3, 0, 2, 0)$ | $(1, 2, 2, 0)$ | $(0, 1, 0, 2)$ | $\{4\}$ | $(Cub)(Con)$ |
| | | | $(3, 0, 2, 0)$ | $(1, 0, 0, 2)$ | $(0, 2, 1, 1)$ | $\{4\}$ | $(Cub)(Con)$ |
| $XII - \{5\}_\square$ | $(1, 1, 1, 1)[4]$ | $(0, 1, 2, 3) + (3, 2, 1, 0)$ | $(1, 0, 3, 0)$ | $(0, 3, 0, 1)$ | $(2, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| $XIII - \{8\}_\square$ | $(1, 1, 2, 3)[7]$ | $(0, 1, 1, 1) + (1, 0, 1, 2)$ | $(3, 2, 1, 0)$ | $(0, 0, 2, 1)$ | $(0, 1, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(1, 0, 3, 0)$ | $(2, 2, 0, 1)$ | $(0, 1, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(4, 3, 0, 0)$ | $(0, 1, 0, 2)$ | $(0, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Con)$ |
| | | | $(3, 2, 1, 0)$ | $(1, 0, 3, 0)$ | $(0, 1, 0, 2)$ | $\{4\}$ | $(Cub)^2$ |
| $XIV - \{6\}_\square$ | $(1, 1, 1, 2)[5]$ | $(0, 1, 1, 2) + (2, 1, 3, 0)$ | $(0, 3, 1, 0)$ | $(3, 0, 0, 2)$ | $(0, 0, 2, 1)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(1, 4, 0, 0)$ | $(3, 0, 0, 2)$ | $(0, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| $XXII - \{7\}_\square$ | $(1, 2, 1, 2)[6]$ | $(0, 1, 1, 2) + (1, 1, 0, 0)$ | $(1, 1, 3, 0)$ | $(0, 2, 0, 1)$ | $(2, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | | $(2, 0, 4, 0)$ | $(2, 0, 0, 2)$ | $(0, 2, 0, 1)$ | $\{4\}$ | $(Qrt)(Con)$ |

Table 9: *The list of K3 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_4$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_4$, $1/2(C_1 + C_2) = E_4$. Here we are present those with $\{7\}_\Delta$ chains [34].*

| N | $\vec{k}_i(\text{eld})$ | Chain | P_1 | P_2 | P_3/C_2 | SIM | $SIM - \text{type}$ |
|------------------------|-------------------------|-------------------------------|----------------|----------------|----------------|---------|---------------------|
| $XV - \{7\}_\Delta$ | $(1, 1, 2, 3)[7]$ | $(1, 2, 3, 0) + (0, 0, 0, 1)$ | $(3, 0, 1, 1)$ | $(0, 3, 0, 1)$ | $(0, 0, 2, 1)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(4, 1, 0, 1)$ | $(0, 3, 0, 1)$ | $(0, 0, 2, 1)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(6, 0, 0, 1)$ | | | $\{6\}$ | M_6 |
| $XVI - \{7\}_\Delta$ | $(1, 1, 2, 4)[8]$ | $(0, 1, 2, 3) + (1, 0, 0, 1)$ | $(1, 3, 0, 1)$ | $(2, 0, 3, 0)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(2, 4, 1, 0)$ | $(2, 0, 3, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(2, 6, 0, 0)$ | | | $\{6\}$ | M_6 |
| $XVII - \{7\}_\Delta$ | $(1, 1, 3, 4)[9]$ | $(0, 1, 2, 3) + (1, 0, 1, 1)$ | $(2, 3, 0, 1)$ | $(0, 0, 3, 0)$ | $(1, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(2, 4, 1, 0)$ | $(0, 0, 3, 0)$ | $(1, 0, 0, 2)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(3, 6, 0, 0)$ | | | $\{6\}$ | M_6 |
| $XVIII - \{7\}_\Delta$ | $(1, 1, 3, 5)[10]$ | $(0, 1, 2, 3) + (1, 0, 1, 2)$ | $(3, 2, 0, 1)$ | $(0, 1, 3, 0)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(4, 3, 1, 0)$ | $(0, 1, 3, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(6, 4, 0, 0)$ | | | $\{6\}$ | M_6 |
| $XIX - \{7\}_\Delta$ | $(1, 1, 4, 6)[12]$ | $(0, 1, 2, 3) + (1, 0, 2, 3)$ | $(3, 3, 0, 1)$ | $(0, 0, 3, 0)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(4, 4, 1, 0)$ | $(0, 0, 3, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(6, 6, 0, 0)$ | | | $\{6\}$ | M_6 |
| $XX - \{7\}_\Delta$ | $(1, 1, 1, 3)[6]$ | $(0, 1, 2, 3) + (2, 1, 0, 3)$ | $(0, 3, 0, 1)$ | $(3, 0, 3, 0)$ | $(0, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(3, 0, 3, 0)$ | $(1, 4, 1, 0)$ | $(0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(0, 6, 0, 0)$ | | | $\{6\}$ | M_6 |
| $XXI - \{7\}_\Delta$ | $(3, 2, 4, 3)[12]$ | $(0, 1, 2, 3) + (3, 1, 2, 0)$ | $(1, 3, 0, 1)$ | $(0, 0, 3, 0)$ | $(2, 0, 0, 2)$ | $\{3\}$ | $(Cub)^2(Con)$ |
| | | | $(0, 4, 1, 0)$ | $(0, 0, 3, 0)$ | $(2, 0, 0, 2)$ | $\{4\}$ | $(Qrt)(Cub)$ |
| | | | $(0, 6, 0, 0)$ | | | $\{6\}$ | M_6 |

$$\{IM\}_4 \mapsto \left(1 \cdot \{4\}_\Delta\right) + \left(2 \cdot \{10\}_\Delta\right)$$

$$\begin{aligned}
& + \left(2 \cdot \{5\}_{\Delta} + 1 \cdot \{5\}_{\square} \right) \\
& + \left(4 \cdot \{9\}_{\Delta} + 2 \cdot \{9\}_{\square} \right) \\
& + \left(7 \cdot \{7\}_{\Delta} + 1 \cdot \{7\}_{\square} \right) \\
& + \left(1 \cdot \{6\}_{\square} \right) + \left(1 \cdot \{8\}_{\square} \right) \\
& \mapsto \{22\}
\end{aligned} \tag{139}$$

This expansion in terms of fibration structures is very helpful for extending these $K3$ results to more general CY_d spaces, via recurrence relations. As we show later, each of the terms $\{10, 4, \dots\}_{\Delta, \square, \dots}$ in the expansion has its own recurrence relation, of which we later derive several examples, indicated in bold script: $\mathbf{2} \cdot \{\mathbf{10}\}_{\Delta}$, etc., providing complete results in any number of dimensions for the numbers of CY_d spaces with these particular fibrations. A similar recurrence formula could be derived for any analogous fibration.

We note that the examples with three cubic monomials, $(Cub)^3$, uniquely determine the chains with intersections $\{10\}_{\Delta}$ and $\{4\}_{\Delta}$, the IMs with two quartic monomials $(Qrt)^2$ and conics C_2 determine the chains with $\{9\}_{\Delta}$ and $\{5\}_{\Delta}$, etc.. The difference between $\{10\}_{\Delta}$ ($\{9\}_{\Delta}$) and $\{4\}_{\Delta}$ ($\{5\}_{\Delta}$) is that the first chains are also determined by other IMs, so one should look more carefully at the structures of the IMs. We make a very important observation in the $K3$ case, namely that each type of IM corresponds to a different type of intersection. For example, the 7 different Weierstrass IMs, $\{3\} \sim (Cub)^2(Con)$ or $\{4\} \sim (Qrt)(Con)$, correspond exactly to the 7 different chains with a $\{7\}_{\Delta}$ intersection. Thus we can expect that knowing the structures of the IMs and their recurrences in higher dimensions n , one can find all the CY_d spaces: $d = n + 2$ which have some particular fibration. We discuss this point further in the next sections.

4.2.3 Applications to CY_3 Spaces and their Fibrations

We now consider here all possible double Diophantine expansions of conic monomials in 5 dimensions, and also expansions in 3 monomials of the unit monomial. Once these triples have been obtained, the extended vectors are obtained as follows. Given one triple of monomials in n dimensions, we look for all possible extended k vectors with all components zero, except for at most three, with the property that the scalar products with the three monomials all give the sum of the k -vector components.

In practice, we form all possible 3×3 matrices obtained by choosing all possible sets of 3 (out of the n) components of the three monomials, and solve the corresponding system

of linear equations, where the constant term is a column vector with all components equal to unity. If the solutions for the 3 unknowns are all positive, and are such that, when multiplied by the determinant of the matrix (to obtain integer values) and summed, they give the determinant itself, then the 3 values (divided if necessary by the greatest common divisor) are the components of a k vector. In case the determinant is null, one looks for solutions among the set of 5 known good solutions $((1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 2, 3))$, and then sets all components to zero, except those three.

Once we have the solutions, i.e., the set of k vectors, we arrange them in sets of $(n - 2)$ to get the chains. Sets that turn out to have the same component null are rejected. We also reject sets that have the following quantity equal to zero: the determinant of the $(n - 2) \times (n - 2)$ matrix obtained by multiplying the $(n - 2) \times n$ matrix of $n - 2$ n -dimensional \vec{k} vectors by its transpose, on the right.

The most time-consuming procedure is that of counting the distinct chains. A permutation of the order of the k vectors and/or a permutation of the order of some components give rise to the same chain. The computation time was reduced by some restrictive cuts on the checks.

In the 5-dimensional case, from the expansion of the unit monomial into 3 monomials, $D_5[1, 1, 1/3]$, we get 116 triples of monomials of maximal degree 3. From the expansion on the second D_r line of the conic monomials in terms of two monomials, $D_5[112/4]$, we get 164 triples of monomials with maximal degree 4. From the first set we get 231 distinct 3-vector chains, whereas from the second one we get 225 different chains. Some chains are present in both sets, and the final number is 259, in complete agreement with the 259 chains found previously by the normal expansion and intersection technique.

We have already mentioned that, among the set of distinct chains, some can be obtained as linear combinations with positive coefficients of other chains. We have therefore looked for the minimal set of irreducible chains that are necessary to get all the others. In this way, we have confirmed that it contains 161 members, as derived previously using the RWV expansion route. Among the 259 reducible chains (161 irreducible chains) there are 11(7) false vectors, corresponding to the minimal choices of the coefficients, $m = n = l = 1$. These false vectors are not reflexive because they are not allowed at the level of arity 2. For example, the IM composed of the cubic monomials $P_1 = (0, 0, 0, 3, 0)$, $P_2 = (0, 3, 2, 0, 1)$ and $P_3 = (3, 0, 1, 0, 2)$ gives the following arity-3 chain: $\vec{k}^{1ex} = (0, 0, 1, 1, 1)$, $\vec{k}^{2ex} = (0, 1, 0, 2, 3)$, $\vec{k}^{3ex} = (3, 0, 1, 0, 2)$. The choice $m = n = l = 1$ corresponds to the vector $\vec{k} = (1, 1, 4, 5, 4)$, which has the following incorrect arity-2 expansion: $(1, 1, 4, 5, 4) = (1, 1, 0, 1, 0) + 4 \cdot (0, 0, 1, 1, 1)$. It follows from this expansion that this vector is outside the arity-2 chain, $m \cdot (1, 1, 0, 1, 0) + n \cdot (0, 0, 1, 1, 1)$, where

the possible coefficients corresponding to the arity-2 chain should satisfy the constraints $m \leq 3$ and $n \leq 3$. The ‘good’ intersection of these vectors $\vec{k}^{ex} = (0, 0, 1, 1, 1)$ and $\vec{k}^{ex} = (1, 1, 0, 1, 0)$ gives a reflexive polyhedron with 30 points and a mirror reflexive polyhedron with 6 points. The corresponding arity-2 chain consists only of the following RWVs:

$$\begin{aligned}
\vec{k}^{eld} &= (1, 1, 1, 2, 1), & m = 1; n = 1, \\
\vec{k}^2 &= (2, 2, 1, 3, 1), & m = 2; n = 1, \\
\vec{k}^3 &= (3, 3, 1, 4, 1), & m = 3; n = 1, \\
\vec{k}^3 &= (3, 3, 2, 5, 2), & m = 3; n = 2.
\end{aligned} \tag{140}$$

Note that the entire list of RWVs in the chain is determined by the structure of the IM. The abovementioned arity-3 chain is normal: for example, the weight vector

$$(3, 2, 2, 5, 9) = (0, 0, 1, 1, 1) + 2 \cdot (0, 1, 0, 2, 3) + (3, 0, 1, 0, 2) \tag{141}$$

with $m = 1, n = 2, l = 1$ is reflexive, as can be checked by Diophantine expansion of one of the cubic monomials $P_2 = (0, 3, 2, 0, 1)$ and the two monomials $M_1 = (0, 6, 0, 0, 1)$ and $M_2 = (0, 0, 4, 0, 1)$ and fixed other monomials from the considered IM.

There are fixed types and numbers of IMs which determine the structures of the full 259 (irreducible 161) chains, and they are similar to those we already indicated for the $K3$ case, as seen, for example, in the following Tables 10 and 11.

$$\begin{aligned}
\{IM\}_5 &\mapsto \left(9 \cdot \{4\}_\Delta + 4 \cdot \{10\}_\Delta \right) \\
&+ \left(16 \cdot \{5\}_\Delta + 5 \cdot \{5\}_\square + 1 \cdot \{5\}_{\square'} \right) \\
&+ \left(11 \cdot \{9\}_\Delta + 5 \cdot \{9\}_\square + 1 \cdot \{9\}_{\square'} \right) \\
&+ \left(28 \cdot \{7\}_\Delta + 7 \cdot \{7\}_\square + 1 \cdot \{7\}_{Quint} \right) \\
&+ \left(8 \cdot \{6\}_\square + 1 \cdot \{6\}_{Quint} \right) \\
&+ \left(6 \cdot \{8\}_\square + 1 \cdot \{8\}_{Quint} \right) \\
&\mapsto \{161\}
\end{aligned} \tag{142}$$

Table 10: *The CY_3 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_5$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_5$, $1/2(C_1 + C_2) = E_5$. Here we present the chains with triangle $\{10 + 4\}$ and $\{9 + 5\}$ intersections [34].*

| σ | $k_i(eld)$ | Chain | P_1 | P_2 | P_3/C_2 | IM | IM - type |
|--------------------|-----------------------|----------------------|-------------------|--------------------|-------------------|-----------|----------------|
| $\{10\}_\Delta$ | $(1, 1, 1, 1, 1)[5]$ | $(0, 0, 0, 0, 1)+$ | $(0, 0, 3, 1, 1)$ | $(0, 3, 0, 1, 1)$ | $(3, 0, 0, 1, 1)$ | $\{3\}$ | $(Cub)^3$ |
| | | $(0, 0, 0, 1, 0)+$ | $(3, 0, 0, 1, 1)$ | $(0, 2, 1, 1, 1)$ | $(0, 1, 2, 1, 1)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | $(1, 1, 1, 0, 0)+$ | $(0, 2, 1, 1, 1)$ | $(1, 0, 2, 1, 1)$ | $(2, 1, 0, 1, 1)$ | $\{3\}$ | $(Con)^3$ |
| | | | $(3, 0, 0, 1, 1)$ | $(1, 0, 2, 1, 1)$ | $(0, 2, 1, 1, 1)$ | $\{4\}$ | $(Cub)(Con)$ |
| $\{4\}_\Delta$ | $(3, 3, 1, 4, 4)[15]$ | $(0, 0, 1, 1, 1)+$ | $(0, 1, 0, 3, 0)$ | $(1, 0, 0, 0, 3)$ | $(2, 2, 3, 0, 0)$ | $\{3\}$ | $(Cub)^3$ |
| | | $(0, 3, 0, 1, 2)+$ | — | — | — | — | — |
| | | $(3, 0, 0, 2, 1)+$ | — | — | — | — | — |
| $\{9\}_\Delta$ | $(1, 1, 1, 1, 1)[5]$ | $(0, 0, 1, 1, 2)+$ | $(0, 0, 0, 4, 0)$ | $(0, 0, 4, 0, 0)$ | $(2, 2, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | $(0, 2, 1, 1, 0)+$ | $(0, 0, 1, 3, 0)$ | $(0, 0, 3, 1, 0)$ | $(2, 2, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | $(2, 0, 1, 1, 0)+$ | $(0, 0, 1, 3, 0)$ | $(1, 1, 2, 0, 1)$ | $(2, 2, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| | | $(1, 0, 0, 0, 1, 2)$ | — | — | — | — | — |
| $\{9\}_{\square'}$ | $(1, 1, 1, 2, 2)[7]$ | $(0, 0, 0, 1, 1)+$ | $(2, 0, 0, 0, 2)$ | $(2, 0, 1, 1, 1)$ | $(0, 2, 0, 0, 2)$ | $\{2\}^4$ | $(Con)^4$ |
| | | $(0, 0, 1, 0, 1)+$ | — | — | — | — | — |
| | | $(1, 1, 0, 0, 0)+$ | — | — | — | — | — |
| $\{5\}_\Delta$ | $(2, 1, 1, 3, 5)[12]$ | $(0, 0, 1, 1, 2)+$ | $(2, 4, 4, 0, 0)$ | $(0, 0, 0, 4, 0)$ | $(1, 0, 0, 0, 2)$ | $\{4\}$ | $(Qrt)^2$ |
| | | $(0, 1, 0, 1, 2)+$ | — | — | — | — | — |
| | | $(2, 0, 0, 1, 1)$ | — | — | — | — | — |
| $\{9\}_{\square}$ | $(1, 1, 2, 1, 5)[10]$ | $(0, 0, 0, 1, 1)+$ | $(4, 4, 0, 2, 0)$ | $(0, 0, 4, 2, 0)$ | $(0, 0, 0, 0, 2)$ | $\{4\}$ | $(Quart)^2$ |
| | | $(0, 1, 1, 0, 2)+$ | $(3, 3, 1, 2, 0)$ | $(1, 1, 3, 2, 0)-$ | $(0, 0, 0, 0, 2)$ | $\{4\}$ | $(Cub)^2$ |
| | | $(1, 0, 1, 0, 2)+$ | $(3, 3, 1, 2, 0)$ | $(0, 0, 2, 1, 1)$ | $(0, 0, 0, 0, 2)$ | $\{3\}$ | $(Cub)(Con)^2$ |
| $\{5\}_{\square}$ | $(3, 3, 4, 5, 3)[19]$ | $(0, 0, 1, 2, 3)+$ | $(1, 0, 0, 3, 0)$ | $(0, 1, 3, 0, 1)$ | $(2, 2, 0, 0, 2)$ | $\{3\}^2$ | $(Cub)^2(Con)$ |
| | | $(0, 3, 1, 2, 0)+$ | — | — | — | — | — |
| | | $(3, 0, 2, 1, 0)+$ | — | — | — | — | — |
| $\{5\}_{\square'}$ | $(2, 1, 2, 1, 2)[8]$ | $(0, 0, 0, 1, 1)+$ | $(0, 0, 2, 0, 2)$ | $(1, 0, 2, 2, 0)$ | $(1, 2, 0, 0, 2)$ | $\{2\}^2$ | $(Con)^2$ |
| | | $(0, 1, 1, 0, 0)+$ | — | — | — | — | — |
| | | $(2, 0, 1, 0, 1)+$ | — | — | — | — | — |

We stress that finding the IMs corresponding to the E_r line is possible in any dimension, because of simple recurrence relations for conic, cubic, quartic and sextic monomials. As an illustration, we discuss the IMs leading to Weierstrass fibrations, which have just the following two structures: $\{3\} \rightarrow (Cub)^2(Con)$ and $\{4\} \rightarrow (Qrt)(Cub)$. There is an additional link between these two types of Weierstrass IMs and the sextic monomials M_6 , as seen in the Tables. As for the case of $K3$ spaces with Weierstrass fibres, here also one can put each IM into correspondence with a sextic monomial M_6 , with the following property: $1/3(M_6 + 2P_2) = C_1$, where $1/2(C_1 + C_2) = E_4$. The last property enables us to find an exact link between Weierstrass IMs and the types of sextic monomials. This link is not one-to-one, but one to i , where $i = p - 1$ depends on the number p of zero components in the monomial M_6 . For instance, the monomial $M_6 = (6, 6, 6, 0, 0)$ has two zero components and, consequently $i = 1$, whereas the monomial $M_6 = (6, 6, 0, 0, 0)$ has three zero components and consequently $i = 2$, and the monomial $M_6 = (6, 0, 0, 0, 0)$ has four zero components and consequently $i = 3$. In the case of the monomial $M_6 = (6, 6, 6, 0, 0)$, the condition $1/3(M_6 + 2P_2) = C_1$ can be satisfied in just one way, with $P_2 = (3, 3, 3, 0, 1)$ and $C_1 = (0, 0, 0, 0, 2)$. However, in the

Table 11: *The CY_3 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_5$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_5$, $1/2(C_1 + C_2) = E_5$. Here we present the chains with box and quintuple $\{8 + 6\}$ intersections and $\{7 + 7\}$ intersections [34].*

| σ | $k_i(eld)$ | Chain | P_1 | P_2 | P_3/C_2 | IM | IM - type |
|-------------------|---------------------|--|---|---|---|-------------------------------------|--|
| $\{8\}_{\square}$ | (1, 1, 1, 3, 5)[11] | (0, 0, 1, 1, 1)+ (0, 1, 0, 1, 2)+ (1, 0, 0, 1, 2)+ | (4, 4, 3, 0, 0) (3, 3, 2, 1, 0) (3, 3, 2, 1, 0) | (0, 0, 1, 0, 2) (1, 1, 0, 3, 0) (0, 0, 1, 0, 2) | (0, 0, 0, 2, 1) (0, 0, 1, 0, 2) (0, 0, 0, 2, 1) | $\{4\}$ $\{4\}$ $\{3\}$ | $(Qrt)(Con)$ $(Cub)(Con)$ $(Cub)(Con)^2$ |
| $\{6\}_{\square}$ | (2, 2, 2, 1, 5)[15] | (0, 0, 0, 1, 1)+ (0, 1, 2, 0, 1)+ (2, 1, 0, 0, 3)+ | (1, 4, 0, 2, 0) (3, 0, 2, 2, 0) — | (3, 0, 2, 2, 0) (0, 3, 1, 0, 1) — | (0, 0, 1, 0, 2) (0, 0, 0, 1, 2) — | $\{4\}$ $\{3\}$ — | $(Qrt)(Cub)$ $(Cub)^2(Con)$ — |
| $\{8\}_{Quint}$ | (1, 1, 2, 1, 1)[6] | (0, 0, 0, 1, 1)+ (0, 1, 1, 0, 0)+ (1, 0, 1, 0, 1) | (3, 2, 0, 2, 0) (2, 2, 0, 1, 1) — | (0, 1, 1, 0, 2) (1, 0, 2, 2, 0) — | (0, 0, 2, 1, 1) (0, 1, 1, 0, 2) — | $\{3\}$ $\{2\}^3$ — | $(Cub)(Con)^2$ $(Con)^3$ — |
| $\{6\}_{Quint}$ | (2, 1, 1, 3, 5)[12] | (0, 0, 1, 1, 1)+ (0, 1, 0, 1, 2)+ (1, 0, 2, 1, 0) | (1, 1, 0, 3, 0) (0, 2, 2, 0, 1) — | (2, 0, 1, 0, 2) (0, 2, 1, 2, 0) — | (0, 2, 2, 0, 1) (2, 0, 1, 0, 2) — | $\{3\}$ $\{2\}^2$ — | $(Cub)(Con)^2$ $(Con)^2$ — |
| $\{7\}_{\square}$ | (2, 2, 2, 1, 3)[10] | (0, 0, 0, 1, 1)+ (0, 1, 2, 0, 1)+ (2, 1, 0, 0, 1)+ | (0, 4, 0, 2, 0) (3, 3, 0, 1, 1) — | (2, 0, 2, 2, 0) (0, 0, 2, 2, 0)– — | (1, 0, 1, 0, 2) (0, 0, 1, 0, 2) — | $\{4\}$ $\{3\}$ — | $(Quart)(Con)$ $(Cub)(Con)^2$ — |
| $\{7\}_{Quint}$ | (1, 2, 3, 1, 2)[19] | (0, 0, 0, 1, 1)+ (0, 1, 1, 0, 1)+ (1, 1, 2, 0, 0)+ | (2, 1, 0, 3, 0) (0, 3, 2, 1, 0) (0, 2, 2, 0, 1) | (1, 0, 1, 0, 2) (2, 0, 0, 2, 1) (1, 0, 1, 0, 2) | (0, 2, 2, 0, 1) (1, 0, 1, 0, 2) (2, 0, 0, 2, 1) | $\{3\}^2$ $\{3\}^2$ $\{2\}^2$ | $(Cub)^2(Con)$ $(Cub)^2(Con)$ $(Con)^2$ |

case of the monomial $M_6 = (6, 6, 0, 0, 0)$ there are already two ways of satisfying the condition $1/3(M_6 + 2P_2) = C_1$, namely $P_2 = (3, 3, 0, 0, 1)$, $C_1 = (0, 0, 0, 0, 2)$ and $P'_2 = (3, 3, 1, 0, 1)$, $C'_1 = (0, 0, 2, 0, 2)$, respectively. The three solutions for the condition $1/3(M_6 + 2P_2) = C_1$ in the case of the monomial $M_6 = (6, 0, 0, 0, 0)$ can be see in Tables 6 and 12 to 16. Using the fact that there exist these multiple examples of Weierstrass IMs for sextic monomials, we obtain the following expression for the number of Weierstrass IMs, $\{3\}$ and $\{4\}$:

$$N_W(n) = C_{n+3}^{n-3} = \frac{(n+3)!}{(6!)(n-3)!}, \quad (143)$$

valid for all dimensions.

Similar recurrence formulae can (in principle) be found for all other types of IMs in arbitrary dimensions n . We now discuss the example of the IM $\{3_{10}\}$. This type of IM is determined by 3 cubic monomials with one important property: the differences between each pair of monomials included in this IM, the combinations $P_i - P_j$, should have coefficients divisible by 3. Therefore, the degrees of these monomials should be constructed only from the numbers 3, 1 and 0. In order to find the recurrence formula, one should look at all possible expansions of the integer numbers $n, n-1, \dots, 3$ in terms of 3 integers. For example, using the compositions of the numbers 3, 4, 5, 6,, respectively, one can get the following results for the number of such IMs, for different dimensions $N(n)$:

$$E_r - line$$

Table 12: *The CY_3 IMs with the Diophantine conditions: $\{3\} : 1/3(P_1 + P_2 + P_3) = E_5$ and $\{4\} : 1/4(P_1 + P_2 + 2C_2) = E_5$, $1/2(C_1 + C_2) = E_5$. Here we start presenting the 28 Weierstrass sets of IMs [34].*

| N | $k_i(eld)$ | Chain | P_1 | P_2 | P_3 | C_2 | IM | IM – type |
|-----|---|---|---|--|------------------------|------------------------|-------------------------------|---|
| 1 | $(1, 1, 1, 6, 9)[18]$ $h_{21} = 272$ $h_{11} = 2$ | $(0, 0, 1, 2, 3) +$ $(0, 1, 0, 2, 3) +$ $(1, 0, 0, 2, 3)$ | $(0, 0, 0, 3, 0)$ $(4, 4, 4, 1, 0)$ $(6, 6, 6, 0, 0)$ | $(3, 3, 3, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 2 | $(1, 1, 1, 5, 8)[16]$ $h_{21} = 231$ $h_{11} = 3$ | $(0, 0, 1, 1, 2) +$ $(0, 1, 0, 2, 3) +$ $(1, 0, 0, 2, 3)$ | $(0, 0, 1, 3, 0)$ $(4, 4, 3, 1, 0)$ $(6, 6, 4, 0, 0)$ | $(3, 3, 2, 0, 1)$ $(0, 0, 1, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 3 | $(1, 1, 1, 5, 7)[15]$ $h_{21} = 208$ $h_{11} = 4$ | $(0, 0, 1, 1, 1) +$ $(0, 1, 0, 2, 3) +$ $(1, 0, 0, 2, 3)$ | $(0, 0, 0, 3, 0)$ $(4, 4, 2, 1, 0)$ $(6, 6, 3, 0, 0)$ | $(3, 3, 2, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(0, 0, 1, 0, 2)$ – | – $(0, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 4 | $(1, 1, 1, 4, 7)[14]$ $h_{21} = 195$ $h_{11} = 3$ | $(0, 0, 1, 0, 1) +$ $(0, 1, 0, 2, 3) +$ $(1, 0, 0, 2, 3)$ | $(0, 0, 2, 3, 0)$ $(4, 4, 2, 1, 0)$ $(6, 6, 2, 0, 0)$ | $(3, 3, 1, 0, 1)$ $(0, 0, 2, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 5 | $(1, 1, 1, 4, 6)[13]$ $h_{21} = 173$ $h_{11} = 5$ | $(0, 0, 1, 0, 0) +$ $(0, 1, 0, 2, 3) +$ $(1, 0, 0, 2, 3)$ | $(0, 0, 1, 3, 0)$ $(4, 4, 1, 1, 0)$ $(6, 6, 1, 0, 0)$ | $(3, 3, 1, 0, 1)$ $(0, 0, 1, 3, 0)$ | $(0, 0, 1, 0, 2)$ – | – $(0, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 6 | $(2, 1, 2, 4, 9)[18]$ $h_{21} = 123$ $h_{11} = 3$ | $(0, 0, 1, 2, 3) +$ $(0, 1, 0, 2, 3) +$ $(2, 0, 1, 0, 3)$ | $(3, 0, 0, 3, 0)$ $(1, 4, 4, 1, 0)$ $(0, 6, 6, 0, 0)$ | $(0, 3, 3, 0, 1)$ $(3, 0, 0, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 7 | $(3, 2, 1, 6, 6)[18]$ $h_{21} = 79$ $h_{11} = 7$ | $(0, 0, 1, 2, 3) +$ $(0, 1, 0, 2, 3) +$ $(3, 0, 1, 2, 0)$ | $(0, 0, 0, 3, 0)$ $(0, 4, 4, 1, 0)$ $(0, 6, 6, 0, 0)$ | $(1, 3, 3, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(2, 0, 0, 0, 2)$ – | – $(2, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |

Table 13: *Continuation of the previous Table.*

| N | $k_i(eld)$ | Chain | P_1 | P_2 | P_3 | C_2 | IM | IM – type |
|-----|---|---|---|--|------------------------|------------------------|-------------------------------|---|
| 8 | $(1, 1, 1, 4, 7)[16]$ $h_{21} = 195$ $h_{11} = 3$ | $(0, 0, 1, 1, 2) +$ $(0, 1, 0, 1, 2) +$ $(1, 0, 0, 2, 3)$ | $(0, 1, 1, 3, 0)$ $(4, 3, 3, 1, 0)$ $(6, 4, 4, 0, 0)$ | $(3, 2, 2, 0, 1)$ $(0, 1, 1, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 9 | $(1, 1, 1, 4, 6)[16]$ $h_{21} = 173$ $h_{11} = 5$ | $(0, 0, 1, 1, 1) +$ $(0, 1, 0, 1, 2) +$ $(1, 0, 0, 2, 3)$ | $(0, 1, 0, 3, 0)$ $(4, 3, 2, 1, 0)$ $(6, 4, 3, 0, 0)$ | $(3, 2, 2, 0, 1)$ $(0, 1, 0, 3, 0)$ | $(0, 0, 1, 0, 2)$ – | – $(0, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 10 | $(1, 1, 1, 3, 6)[16]$ $h_{21} = 165$ $h_{11} = 3$ | $(0, 0, 1, 0, 1) +$ $(0, 1, 0, 1, 2) +$ $(1, 0, 0, 2, 3)$ | $(0, 1, 2, 3, 0)$ $(4, 3, 2, 1, 0)$ $(6, 4, 2, 0, 0)$ | $(3, 2, 1, 0, 1)$ $(0, 1, 2, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 11 | $(1, 1, 1, 3, 5)[16]$ $h_{21} = 144$ $h_{11} = 4$ | $(0, 0, 1, 0, 0) +$ $(0, 1, 0, 1, 2) +$ $(1, 0, 0, 2, 3)$ | $(0, 1, 1, 3, 0)$ $(4, 3, 1, 1, 0)$ $(6, 4, 1, 0, 0)$ | $(3, 2, 1, 0, 1)$ $(0, 1, 1, 3, 0)$ | $(0, 0, 1, 0, 2)$ – | – $(0, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 12 | $(1, 2, 2, 4, 5)[14]$ $h_{21} = 83$ $h_{11} = 5$ | $(0, 0, 1, 1, 2) +$ $(0, 2, 1, 1, 0) +$ $(1, 0, 0, 2, 3)$ | $(0, 0, 1, 3, 0)$ $(4, 0, 3, 1, 0)$ $(6, 0, 4, 0, 0)$ | $(3, 1, 2, 0, 1)$ $(0, 0, 1, 3, 0)$ | $(0, 2, 0, 0, 2)$ – | – $(0, 2, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 12' | $(3, 2, 1, 5, 5)[16]$ $h_{21} = 67$ $h_{11} = 9$ | $(0, 0, 1, 1, 2) +$ $(0, 1, 0, 2, 3) +$ $(3, 1, 0, 2, 0)$ | $(0, 0, 1, 3, 0)$ $(4, 0, 3, 1, 0)$ $(6, 0, 4, 0, 0)$ | $(3, 1, 2, 0, 1)$ $(0, 0, 1, 3, 0)$ | $(0, 2, 0, 0, 2)$ – | – $(0, 2, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 13 | $(1, 2, 4, 2, 5)[14]$ $h_{21} = 83$ $h_{11} = 5$ | $(0, 0, 1, 1, 2) +$ $(0, 2, 1, 1, 0) +$ $(1, 0, 2, 0, 3)$ | $(0, 3, 3, 1, 0)$ $(4, 1, 1, 3, 0)$ $(6, 0, 0, 4, 0)$ | $(3, 0, 0, 2, 1)$ $(0, 3, 3, 1, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 13' | $(2, 2, 1, 3, 8)[16]$ $h_{21} = 103$ $h_{11} = 3$ | $(0, 0, 1, 1, 2) +$ $(0, 1, 0, 2, 3) +$ $(2, 1, 0, 0, 3)$ | $(0, 0, 1, 3, 0)$ $(4, 0, 3, 1, 0)$ $(6, 0, 0, 4, 0)$ | $(3, 1, 2, 0, 1)$ $(0, 0, 1, 3, 0)$ | $(0, 0, 0, 0, 2)$ – | – $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |

$$3 = 1 + 1 + 1,$$

Table 14: *Continuation of the previous Table.*

| N | $\vec{k}_i(eld)$ | $Chain$ | P_1 | P_2 | P_3 | C_2 | IM | $IM - type$ |
|-----|---|---|---|--|------------------------|------------------------|-------------------------------|---|
| 14 | $(1, 1, 1, 4, 5)[15]$ $h_{11} = 154$ $h_{11} = 4$ | $(0, 0, 1, 1, 1)+$ $(0, 1, 0, 1, 1)+$ $(1, 0, 0, 2, 3)$ | $(0, 0, 0, 3, 0)$ $(4, 2, 2, 1, 0)$ $(6, 3, 3, 0, 0)$ | $(3, 2, 2, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(0, 0, 0, 0, 2)$ — | — $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 15 | $(1, 1, 1, 3, 5)[11]$ $h_{21} = 144$ $h_{11} = 4$ | $(0, 0, 1, 0, 1)+$ $(0, 1, 0, 1, 1)+$ $(1, 0, 0, 2, 3)$ | $(0, 0, 2, 3, 0)$ $(4, 2, 2, 1, 0)$ $(6, 3, 2, 0, 0)$ | $(3, 2, 1, 0, 1)$ $(0, 0, 2, 3, 0)$ | $(0, 1, 0, 0, 2)$ — | — $(0, 1, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 16 | $(1, 1, 1, 3, 4)[10]$ $h_{21} = 126$ $h_{11} = 4$ | $(0, 0, 1, 0, 0)+$ $(0, 1, 0, 1, 1)+$ $(1, 0, 0, 2, 3)$ | $(0, 0, 1, 3, 0)$ $(4, 2, 1, 1, 0)$ $(6, 3, 1, 0, 0)$ | $(3, 2, 1, 0, 1)$ $(0, 0, 1, 3, 0)$ | $(0, 1, 1, 0, 2)$ — | — $(0, 1, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 17 | $(3, 2, 1, 5, 4)[15]$ $h_{21} = 66$ $h_{11} = 3$ | $(0, 0, 1, 1, 1)+$ $(0, 1, 0, 2, 3)+$ $(3, 1, 0, 2, 0)$ | $(0, 0, 0, 3, 0)$ $(0, 4, 2, 1, 0)$ $(0, 6, 3, 0, 0)$ | $(1, 3, 2, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(2, 0, 1, 0, 2)$ — | — $(2, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 17' | $(1, 1, 4, 2, 4)[15]$ $h_{21} = 101$ $h_{11} = 5$ | $(0, 0, 1, 1, 1)+$ $(0, 1, 0, 1, 1)+$ $(1, 0, 3, 0, 2)$ | $(0, 0, 0, 3, 0)$ $(0, 4, 2, 1, 0)$ $(0, 6, 3, 0, 0)$ | $(1, 3, 2, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(2, 0, 1, 0, 2)$ — | — $(2, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 18 | $(2, 2, 1, 3, 7)[15]$ $h_{21} = 89$ $h_{11} = 5$ | $(0, 0, 1, 1, 1)+$ $(0, 1, 0, 2, 3)+$ $(2, 1, 0, 0, 3)$ | $(3, 0, 0, 3, 0)$ $(1, 4, 2, 1, 0)$ $(0, 6, 3, 0, 0)$ | $(0, 3, 2, 0, 1)$ $(3, 0, 0, 3, 0)$ | $(0, 0, 1, 0, 2)$ — | — $(0, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 18' | $(1, 1, 3, 2, 5)[12]$ $h_{21} = 105$ $h_{11} = 3$ | $(0, 0, 1, 1, 1)+$ $(0, 1, 0, 1, 1)+$ $(1, 0, 2, 0, 3)$ | $(3, 0, 0, 3, 0)$ $(1, 4, 2, 1, 0)$ $(0, 6, 3, 0, 0)$ | $(0, 3, 2, 0, 1)$ $(3, 0, 0, 3, 0)$ | $(0, 0, 1, 0, 2)$ — | — $(0, 0, 1, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |

Table 15: *Continuation of the previous Table.*

| N | $\vec{k}_i(eld)$ | $Chain$ | P_1 | P_2 | P_3 | C_2 | IM | $IM - type$ |
|-------|---|---|---|--|------------------------|------------------------|-------------------------------|---|
| 19 | $(1, 2, 1, 1, 5)[10]$ $h_{21} = 145$ $h_{11} = 1$ | $(0, 0, 0, 1, 1)+$ $(0, 0, 1, 0, 1)+$ $(1, 2, 0, 0, 3)$ | $(0, 3, 2, 2, 0)$ $(4, 1, 2, 2, 0)$ $(6, 0, 2, 2, 0)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 2, 2, 0)$ | $(0, 0, 0, 0, 2)$ — | — $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 20 | $(1, 2, 1, 1, 4)[9]$ $h_{21} = 123$ $h_{11} = 3$ | $(0, 0, 0, 1, 0)+$ $(0, 0, 1, 0, 1)+$ $(1, 2, 0, 0, 3)$ | $(0, 3, 2, 1, 0)$ $(4, 1, 2, 1, 0)$ $(6, 0, 2, 1, 0)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 2, 1, 0)$ | $(0, 0, 0, 1, 2)$ — | — $(0, 0, 0, 1, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 21 | $(1, 2, 1, 2, 4)[9]$ $h_{21} = 99$ $h_{11} = 3$ | $(0, 0, 0, 1, 1)+$ $(0, 0, 1, 1, 0)+$ $(1, 2, 0, 0, 3)$ | $(0, 3, 0, 2, 0)$ $(4, 1, 0, 2, 0)$ $(6, 0, 0, 2, 0)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 0, 2, 0)$ | $(0, 0, 2, 0, 2)$ — | — $(0, 0, 2, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 21' | $(2, 4, 3, 1, 4)[14]$ $h_{21} = 64$ $h_{11} = 6$ | $(0, 0, 0, 1, 1)+$ $(1, 2, 0, 0, 3)+$ $(1, 2, 3, 0, 0)$ | $(0, 3, 0, 2, 0)$ $(4, 1, 0, 2, 0)$ $(6, 0, 0, 2, 0)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 0, 2, 0)$ | $(0, 0, 2, 0, 2)$ — | — $(0, 0, 2, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 21'' | $(2, 4, 5, 3, 4)[18]$ $h_{21} = 39$ $h_{11} = 7$ | $(1, 2, 3, 0, 0)+$ $(1, 2, 0, 0, 3)+$ $(0, 0, 2, 3, 1)$ | $(0, 3, 0, 2, 0)$ $(4, 1, 0, 2, 0)$ $(6, 0, 0, 2, 0)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 0, 2, 0)$ | $(0, 0, 2, 0, 2)$ — | — $(0, 0, 2, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 21''' | $(1, 2, 2, 1, 2)[9]$ $h_{21} = 86$ $h_{11} = 2$ | $(1, 2, 3, 0, 0)+$ $(1, 2, 0, 0, 3)+$ $(0, 0, 1, 2, 1)$ | $(0, 3, 0, 2, 0)$ $(4, 1, 0, 2, 0)$ $(6, 0, 0, 2, 0)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 0, 2, 0)$ | $(0, 0, 2, 0, 2)$ — | — $(0, 0, 2, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 22 | $(2, 2, 2, 1, 7)[9]$ $h_{21} = 122$ $h_{11} = 2$ | $(0, 0, 0, 1, 1)+$ $(0, 1, 2, 0, 3)+$ $(2, 1, 0, 0, 3)$ | $(3, 0, 3, 2, 0)$ $(1, 4, 1, 2, 0)$ $(0, 6, 0, 2, 0)$ | $(0, 3, 0, 1, 1)$ $(3, 0, 3, 2, 0)$ | $(0, 0, 0, 0, 2)$ — | — $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |

$$4 = 2 + 1 + 1,$$

$$N(4) = 2$$

.....

$$5 = 3 + 1 + 1, \quad 2 + 2 + 1$$

$$N(5) = 2N(4) + 2 = 4$$

Table 16: *Continuation of the previous Table.*

| N | $\vec{k}_i(eld)$ | $Chain$ | P_1 | P_2 | P_3 | C_2 | IM | $IM - type$ |
|-----|--|---|---|--|------------------------|------------------------|-------------------------------|---|
| 23 | $(1, 2, 3, 1, 1)[8]$ $h_{21} = 106$ $h_{11} = 2$ | $(0, 0, 0, 0, 1,) +$ $(0, 0, 0, 1, 0) +$ $(1, 2, 3, 0, 0)$ | $(0, 3, 0, 1, 1)$ $(4, 1, 0, 1, 1)$ $(6, 0, 0, 1, 1)$ | $(3, 0, 1, 1, 1)$ $(0, 3, 0, 1, 1)$ | $(0, 0, 2, 1, 1)$ — | — $(0, 0, 2, 1, 1)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 24 | $(3, 2, 4, 3, 1)[13]$ $h_{21} = 62$ $h_{11} = 5$ | $(0, 0, 0, 0, 1,) +$ $(0, 1, 2, 3, 0) +$ $(3, 1, 2, 0, 0)$ | $(0, 0, 3, 0, 1)$ $(0, 4, 1, 0, 1)$ $(0, 6, 0, 0, 1)$ | $(1, 3, 0, 1, 1)$ $(0, 0, 3, 1, 1)$ | $(2, 0, 0, 2, 1)$ — | — $(2, 0, 0, 2, 1)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 25 | $(3, 2, 4, 6, 1)[16]$ $h_{21} = 73$ $h_{11} = 5$ | $(0, 0, 0, 0, 1,) +$ $(0, 1, 2, 3, 0) +$ $(2, 1, 0, 3, 0)$ | $(0, 3, 0, 1, 1)$ $(1, 4, 1, 0, 1)$ $(0, 6, 0, 0, 1)$ | $(3, 0, 3, 0, 1)$ $(3, 0, 3, 0, 1)$ | $(0, 0, 0, 2, 1)$ — | — $(0, 0, 0, 2, 1)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 26 | $(1, 1, 1, 2, 1)[6'']$ $h_{21} = 103$ $h_{11} = 1$ | $(0, 0, 1, 2, 3) +$ $(0, 3, 1, 2, 0) +$ $(3, 0, 1, 2, 0)$ | $(0, 0, 0, 3, 0)$ $(0, 0, 4, 1, 0)$ $(0, 0, 6, 0, 0)$ | $(1, 1, 3, 0, 1)$ $(0, 0, 0, 3, 0)$ | $(2, 2, 0, 0, 2)$ — | — $(2, 2, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 27 | $(3, 2, 3, 4, 6)[18]$ $h_{21} = 53$ $h_{11} = 5$ | $(0, 0, 1, 2, 3) +$ $(0, 2, 1, 0, 3) +$ $(3, 0, 1, 2, 0)$ | $(0, 3, 0, 3, 0)$ $(0, 1, 4, 1, 0)$ $(0, 0, 6, 0, 0)$ | $(1, 0, 3, 0, 1)$ $(0, 3, 0, 3, 0)$ | $(2, 0, 0, 0, 2)$ — | — $(2, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |
| 28 | $(2, 2, 3, 2, 9)[18]$ $h_{21} = 112$ $h_{11} = 4$ | $(0, 0, 1, 2, 3) +$ $(0, 2, 1, 0, 3) +$ $(2, 0, 1, 0, 3)$ | $(3, 3, 0, 3, 0)$ $(1, 1, 4, 1, 0)$ $(0, 0, 6, 0, 0)$ | $(0, 0, 3, 0, 1)$ $(3, 3, 0, 3, 0)$ | $(0, 0, 0, 0, 2)$ — | — $(0, 0, 0, 0, 2)$ | $\{3\}$ $\{4\}$ $\{6\}$ | $(Cub)^2(Con)$ $(Qrt)(Cub)$ M_6 |

$$6 = 4 + 1 + 1, \quad 3 + 2 + 1, \quad 2 + 2 + 2$$

$$N(6) = N(4) + 3 = 7$$

.....

$$7 = 5 + 1 + 1, \quad 4 + 2 + 1, \quad 3 + 3 + 1$$

$$= 3 + 2 + 2$$

$$N(7) = N(6) + 4 = 11,$$

.....

(144)

Take into account that for the case $n = 3$, one has only the following combination of monomials: $(3, 0, 0,)$, $(0, 3, 0)$, $(0, 0, 3)$, which corresponds in our description to the expansion of the number 3 in three units. So, for the $K3$ case with $n = 4$, we already have one $\{IM\} = (3, 0, 0, 1), (0, 3, 0, 1), (0, 0, 3, 1)$, which again corresponds to the expansion of the number 3 in terms of three units. However, there also appears another possibility: $\{IM\} = (3, 3, 0, 0), (0, 0, 3, 0), (0, 0, 0, 3)$, which corresponds to the unique expansion of the number 4 in terms of integer positive numbers. For the CY_3 case with $n = 5$, there are the following IMs:

$$\{IM\} = (3, 0, 0, 1, 1), (0, 3, 0, 1, 1), (0, 0, 3, 1, 1); \quad \rightarrow 3(3) = 1 + 1 + 1$$

$$\{IM\} = (3, 3, 0, 0, 1), (0, 0, 3, 0, 1), (0, 0, 0, 3, 1); \quad \rightarrow 4(3) = 2 + 1 + 1$$

$$N(4) = 2 \tag{145}$$

$$\begin{aligned} \{IM\} &= (3, 3, 3, 0, 0), (0, 0, 0, 3, 0), (0, 0, 0, 0, 3); & \rightarrow 5(3) = 3 + 1 + 1 \\ \{IM\} &= (3, 3, 0, 0, 0), (0, 0, 3, 3, 0), (0, 0, 0, 0, 3); & \rightarrow 5(3) = 2 + 2 + 1 \\ N(5) &= N(4) + 2 = 4, \end{aligned} \tag{146}$$

where the last two IMs correspond to the two expansions of the number 5. This recurrence can easily be continued to higher dimensions $n > 5$. To get the total number of IMs $\{10\}_\Delta$ in dimension n , one should sum over all the numbers characterizing the possible decompositions: $n, n-1, \dots, 3$. For example, for CY_d : $d+2 = n = 4, 5, 6, 7, \dots$, these numbers are 2, 4, 7, 11, ... (see Figure 8).

Similarly, one can find a recurrence relation for $\{IM\} = \{9\}_\Delta$, which is constructed from two quartic monomials, one conic and E_n . In this case the difference of the two quartic monomials should be divisible by four. Taking into account all possible quartics, after some effort, one can find the following formula for the number of these IMs:

$$N_{9\Delta} = \frac{1}{3} \cdot (n-2)(n^2 - 4n + 6). \tag{147}$$

This expression gives the following numbers: 1, 4, 11, 24, 45, 76, 119, 176, 249, ... for $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, \dots$, respectively.

4.2.4 Applications to CY_4 Spaces

In the 6-dimensional case, we have, analogously to the previous case, two sets of triples of monomials: one of 377 and the other of 494. From the first set we get 5,216 different 4-vector chains and from the second one we get 4953 chains. The union set has 5527 members, consistent with the 5,607 that we got by the ‘expansion and intersection’ method. As in the 5-dimensional case, the difference can be made up if we include the IMs with double conic monomials.

A further reduction in the number of chains has to be considered, from the 5,607 6-dimensional 4-vector chains to 2111 independent chains. We have already mentioned that there are different types of IMs even among the cubics $\{3\}$ and quartics $\{4\}$, and the number of different conics grows monotonically with increasing dimension n . We have also already remarked that there exists a recurrence formula for all types of IMs with arbitrary dimension n , and have already discussed the recurrences of the Weierstrass IMs $\{3_W\}$ and $\{4_W\}$. The possible types of cubic $\{3\}$, quartic $\{4\}$ and double conic IMs which describe the 2111

irreducible CY_3 chains have different structures, corresponding to the different types of intersections, that we can illustrate by the following expression, see also Tables 17, 18 and 19:

$$\begin{aligned}
\{IM\}_6 &\mapsto \left(37 \cdot \{4\}_\Delta + \mathbf{7} \cdot \{10\}_\Delta \right) \\
&+ \left(66 \cdot \{5\}_\Delta + 27 \cdot \{5\}_\square + 6 \cdot \{5\}_{\square'} \right) \\
&+ \left(\mathbf{24} \cdot \{9\}_\Delta + 11 \cdot \{9\}_\square + 5 \cdot \{9\}_{\square'} \right) \\
&+ \left(\mathbf{84} \cdot \{7\}_\Delta + 28 \cdot \{7\}_\square + 5 \cdot \{7\}_{Quint} + 1 \cdot \{7\}_{Sext} \right) \\
&+ \left(36 \cdot \{6\}_\square + 5 \cdot \{6\}_{Quint} \right) \\
&+ \left(21 \cdot \{8\}_\square + 5 \cdot \{8\}_{Quint} \right) \\
&\mapsto \{2111\}
\end{aligned} \tag{148}$$

The recurrence relation for Calabi-Yau spaces with elliptic fibres $\{10\}_\Delta$ can be extended to the cases of CY_d spaces with $K3$ fibres, described by $\vec{k}_4 = (1, 1, 1, 1)[4]$, whose algebraic equation includes the 35-point monomial and its mirror with 5 points. The IM_4 for this $K3$ space contains the four quartic monomials P_1, P_2, P_3, P_4 obeying the Diophantine equation: $(P_1 + P_2 + P_3 + P_4)/4 = E_4$. These monomials have in addition one very important condition: $P_i - P_j$ should be divisible by 4 for each choice of $i, j = 1, 2, 3, 4, i \neq j$. The types of different n -dimensional $\{IM\}_4$, describing the $CY_d : n = d + 2 \geq 4$ spaces with $\{35\}_\Delta$ fibres are constructed only from the numbers 4 and 0. The number 1 will play an additional role. Therefore, similarly to the case of the third E_r line, the recurrence formulae for these IMs will be determined from the expansions of positive integer numbers in terms of four positive integers, i.e., (see Figure 8),

$$\begin{aligned}
&K3 - line \\
n = 4(4) &= 1 + 1 + 1 + 1 \\
n = 5(4) &= 2 + 1 + 1 + 1 \\
&N(5) = 2 \\
&..... \\
n = 6(4) &= 3 + 1 + 1 + 1 = 2 + 2 + 1 + 1 \\
&N(6) = N(5) + 2 = 4
\end{aligned}$$

$$\begin{aligned}
& \dots\dots\dots \\
n = 7(4) &= 4 + 1 + 1 + 1 = 3 + 2 + 1 + 1 = 2 + 2 + 2 + 1 \\
& N(7) = N(6) + 3 = 7
\end{aligned}$$

$$\begin{aligned}
& \dots\dots\dots \\
n = 8(4) &= 5 + 1 + 1 + 1 = 4 + 2 + 1 + 1 = 3 + 3 + 1 + 1 = \\
&= 3 + 2 + 2 + 1 = 2 + 2 + 2 + 2 \\
& N(8) = N(7) + 5 = 12
\end{aligned}$$

$$\begin{aligned}
& \dots\dots\dots \\
n = 9(4) &= 6 + 1 + 1 + 1 = 5 + 2 + 1 + 1 = 4 + 3 + 1 + 1 = \\
&= 4 + 2 + 2 + 1 = 3 + 3 + 2 + 1 = 3 + 2 + 2 + 2 \\
& N(9) = N(8) + 6 = 18
\end{aligned}$$

$$\begin{aligned}
& \dots\dots\dots \\
n = 10(4) &= 7 + 1 + 1 + 1 = 6 + 2 + 1 + 1 = 5 + 3 + 1 + 1 = 5 + 2 + 2 + 1 = \\
&= 4 + 4 + 1 + 1 = 4 + 3 + 2 + 1 = 3 + 3 + 3 + 1 = 3 + 3 + 2 + 2 \\
& N(10) = N(9) + 8 = 26
\end{aligned}$$

$$\begin{aligned}
& \dots\dots\dots \\
n = 11(4) &= 8 + 1 + 1 + 1 = 7 + 2 + 1 + 1 = 6 + 3 + 1 + 1 = 6 + 2 + 2 + 1 = \\
&= 5 + 4 + 1 + 1 = 5 + 3 + 2 + 1 = 5 + 2 + 2 + 2 = 4 + 4 + 2 + 1 \\
&= 4 + 3 + 3 + 1 = 4 + 3 + 2 + 2 = 3 + 3 + 3 + 2 \\
& N(11) = N(10) + 11 = 37
\end{aligned}$$

(149)

Thus, the corresponding numbers of these four $\{IMs\}$ with dimension n are equal to sums of the possible expansions of the integers indicated in this expression, i.e., for dimensions 5, 6, 7, 8, 9, 10, 11, they are equal to 2, 4, 7, 12, 18, 26, 37, ..., respectively.

Similarly, this example can be extended to CY_4 (CY_d) spaces with a CY_3 (CY_{d-1}) fibre described by the RWV $\vec{k}_5 = (1, 1, 1, 1, 1)[5]$ ($\vec{k}_n = (1, \dots, 1)_n$). The results for the types of the corresponding IMs can be obtained from the following expansions (see also Fig. 8):

$CY_3 - line$

$$n = 5(5) = 1 + 1 + 1 + 1 + 1$$

$$\begin{aligned}
n = 6(5) &= 2 + 1 + 1 + 1 + 1 \\
&N(6) = 2 \\
&\dots\dots\dots \\
n = 7(5) &= 3 + 1 + 1 + 1 + 1 = 2 + 2 + 1 + 1 + 1 \\
&N(7) = N(6) + 2 = 4 \\
&\dots\dots\dots \\
n = 8(5) &= 4 + 1 + 1 + 1 + 1 = 3 + 2 + 1 + 1 + 1 = 2 + 2 + 2 + 1 + 1 \\
&N(8) = N(7) + 3 = 7 \\
&\dots\dots\dots \\
n = 9(5) &= 5 + 1 + 1 + 1 + 1 = 4 + 2 + 1 + 1 + 1 = 3 + 3 + 1 + 1 + 1 \\
&= 3 + 2 + 2 + 1 + 1 = 2 + 2 + 2 + 2 + 1 \\
&N(9) = N(8) + 5 = 12 \\
&\dots\dots\dots \\
n = 10(5) &= 6 + 1 + 1 + 1 + 1 = 5 + 2 + 1 + 1 + 1 = 4 + 3 + 1 + 1 + 1 \\
&= 4 + 2 + 2 + 1 + 1 = 3 + 3 + 2 + 1 + 1 = 3 + 2 + 2 + 2 + 1 = 2 + 2 + 2 + 2 + 2 \\
&N(10) = N(9) + 7 = 19 \\
&\dots\dots\dots \\
n = 11(5) &= 7 + 1 + 1 + 1 + 1 = 6 + 2 + 1 + 1 + 1 = 5 + 3 + 1 + 1 + 1 \\
&= 5 + 2 + 2 + 1 + 1 = 4 + 4 + 1 + 1 + 1 = 4 + 3 + 2 + 1 + 1 = 4 + 2 + 2 + 2 + 1 \\
&= 3 + 3 + 3 + 1 + 1 = 3 + 3 + 2 + 2 + 1 = 3 + 2 + 2 + 2 + 2 \\
&N(11) = N(10) + 10 = 29 \\
&\dots\dots\dots
\end{aligned} \tag{150}$$

The same approach can clearly be extended to establish the numbers of any other desired IMs.

5 Summary

We have presented a Universal Calabi-Yau Algebra (UCYA) which provides a two-parameter classification of $CY - d$ spaces in terms of arity and dimension. This algebra is based on the following ingredients:

Table 17: *The CY_4 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_6$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_6$, $1/2(C_1 + C_2) = E_6$. Here we present the chains with triangle $\{10 + 4\}$ and $\{9 + 5\}$ intersections.*

| σ | $k_i(eld)$ | Chain | P_1 | P_2 | P_3/C_2 | IM | IM - type |
|-----------------|------------------------|--|--|--|--|--|--|
| $\{10\}_\Delta$ | (1, 1, 1, 1, 1)[6] | (0, 0, 1, 0, 1, 1)+ (0, 1, 0, 0, 0, 0)+ (0, 0, 0, 1, 0, 0)+ (1, 0, 0, 0, 1, 1)+ | (0, 1, 1, 1, 0, 3) (3, 1, 1, 1, 0, 0) (2, 1, 1, 1, 0, 1) (3, 1, 1, 1, 0, 0) | (0, 1, 1, 1, 3, 0) (0, 1, 1, 1, 1, 2) (1, 1, 1, 1, 2, 0) (1, 1, 1, 1, 0, 2) | (3, 1, 1, 1, 0, 0) (0, 1, 1, 1, 2, 1) (0, 1, 1, 1, 1, 2) (0, 1, 1, 1, 2, 1) | $\{3\}$ $\{3\}$ $\{3\}$ $\{4\}$ | $(Cub)^3$ $(Cub)(Con)^2$ $(Con)^3$ $(Cub)(Con)$ |
| $\{4\}_\Delta$ | (3, 1, 3, 4, 4, 2)[17] | (0, 1, 0, 0, 1, 0)+ (2, 0, 0, 1, 3, 0)+ (0, 0, 3, 2, 0, 1)+ (0, 0, 0, 1, 0, 0) | (0, 0, 1, 0, 2, 3) — — — | (0, 1, 0, 3, 1, 0) — — — | (3, 2, 2, 0, 0, 0) — — — | $\{3\}$ — — — | $(Cub)^3$ — — — |
| $\{9\}_\Delta$ | (1, 1, 1, 1, 1, 2)[7] | (0, 0, 1, 0, 1, 0)+ (0, 1, 0, 0, 0, 0)+ (0, 0, 0, 1, 0, 0)+ (1, 0, 0, 0, 1, 2) | (0, 4, 1, 1, 0, 0) (1, 3, 1, 1, 0, 0) (3, 1, 1, 1, 0, 0) — | (4, 0, 1, 1, 0, 0) (3, 1, 1, 1, 0, 0) (0, 2, 1, 1, 1, 1) — | (0, 0, 1, 1, 2, 2) (0, 0, 1, 1, 1, 2) (0, 0, 1, 1, 2, 2) — | $\{4\}$ $\{4\}$ $\{3\}$ — | $(Qrt)^2$ $(Qrt)^2$ $(Cub)(Con)^2$ — |
| $\{5\}_\Delta$ | (1, 1, 1, 2, 3, 6)[14] | (0, 0, 1, 0, 0, 1)+ (1, 0, 0, 0, 1, 2)+ (0, 1, 0, 0, 1, 2)+ (0, 0, 0, 2, 1, 1) | (4, 4, 2, 2, 0, 0) — — — | (0, 0, 2, 0, 4, 0) — — — | (0, 0, 0, 1, 0, 2) — — — | $\{4\}$ — — — | $(Qrt)^2$ — — — |
| $\{9\}_\square$ | (2, 1, 1, 3, 2, 2)[11] | (1, 0, 0, 1, 1, 0)+ (0, 1, 0, 1, 1, 0)+ (0, 0, 1, 0, 0, 1)+ (1, 0, 0, 1, 0, 1) | (3, 3, 2, 0, 0, 0) (3, 3, 2, 0, 0, 0)— (2, 2, 2, 1, 0, 0) — | (0, 0, 1, 2, 1, 1) (1, 1, 2, 0, 2, 2)— (2, 2, 1, 0, 1, 1) — | (0, 0, 0, 1, 2, 2) (0, 0, 0, 2, 1, 1)— (1, 1, 0, 0, 2, 2) — | $\{3\}$ $\{4\}$ $\{2\}^3$ — | $(Cub)(Con)^2$ $(Cub)(Con)$ $(Con)^3$ — |
| $\{9\}_{Quadr}$ | (1, 1, 1, 1, 2, 1)[7] | (0, 0, 0, 0, 0, 1)+ (0, 0, 0, 1, 1, 0)+ (0, 0, 1, 0, 1, 0)+ (1, 1, 0, 0, 0, 0) | (0, 2, 0, 0, 2, 1) — — — | (0, 2, 2, 2, 0, 1) — — — | (1, 1, 2, 2, 0, 1) — — — | $\{2\}^4$ — — — | $(Con)^4$ — — — |
| $\{5\}_\square$ | (3, 2, 1, 3, 4, 6)[19] | (0, 0, 1, 0, 0, 0)+ (1, 0, 0, 3, 2, 0)+ (2, 0, 0, 0, 1, 3)+ (0, 2, 0, 0, 1, 3) | (3, 3, 1, 1, 0, 0) — — — | (0, 0, 1, 0, 3, 1) — — — | (0, 0, 1, 2, 0, 2) — — — | $\{3\}^2$ — — — | $(Cub)^2(Con)$ — — — |
| $\{5\}_{Quadr}$ | (1, 1, 1, 1, 1, 2)[7] | (0, 0, 0, 0, 1, 1)+ (0, 0, 1, 1, 0, 0)+ (0, 2, 0, 1, 0, 1)+ (2, 0, 0, 1, 1, 0) | (0, 1, 0, 2, 2, 0) — — — | (1, 0, 0, 2, 0, 2) — — — | (1, 2, 2, 0, 2, 0) — — — | $\{2\}^2$ — — — | $(Con)^2$ — — — |

- Universal composition rules
- Normal expansions and Diophantine decompositions
- Mirror symmetry

We have shown that this algebraic approach leads us to a natural formalism for a unified description of complex geometry in all dimensions, including $K3$ spaces and Calabi-Yau d -folds for any d .

Since the description of the UCYA is based on structures with two integer parameters, the arity and dimension of the reflexive weight vectors (RWVs), we have classified the structures of CY_d spaces along the diagonal A_r, D_r, E_r, \dots lines in this plane. In this article we have studied only the d -folds along the first three lines, presenting new results for low d and some

Table 18: *Continuation of the previous Table.*

| σ | $\vec{k}_i(eld)$ | Chain | P_1 | P_2 | P_3/C_2 | IM | IM – type |
|-------------------|------------------------|---|---|---|---|-------------------------------------|--|
| $\{8\}_{\square}$ | (1, 1, 1, 1, 2, 5)[11] | (0, 0, 1, 0, 0, 1)+ (0, 0, 0, 1, 0, 1)+ (1, 0, 0, 0, 1, 2)+ (0, 1, 0, 0, 1, 1) | (4, 3, 2, 2, 0, 0) (3, 2, 2, 2, 1, 0) — — | (0, 1, 0, 0, 0, 2) (0, 1, 0, 0, 0, 2) — — | (0, 0, 1, 1, 2, 1) (0, 0, 1, 1, 2, 1) — — | {4} {3} — — | (Qrt)(Cub) (Cub)(Con) ² — — |
| $\{8\}_{Quint}$ | (2, 1, 1, 3, 2, 1)[10] | (0, 0, 1, 1, 0, 0)+ (0, 1, 0, 0, 1, 0)+ (1, 0, 0, 1, 1, 0)+ (1, 0, 0, 1, 0, 1) | (3, 2, 2, 0, 0, 0) (3, 2, 2, 2, 0, 0) (0, 0, 1, 1, 2, 2) — | (0, 1, 0, 2, 1, 1) (1, 2, 0, 2, 0, 0) (0, 1, 0, 2, 1, 1) — | (0, 0, 1, 1, 2, 2) (0, 0, 1, 0, 2, 2) (1, 0, 2, 0, 2, 2) — | {3} {4} {2} ³ — | (Cub)(Con) ² (Cub)(Con) (Con) ³ — |
| $\{6\}_{\square}$ | (2, 4, 1, 3, 4, 1)[15] | (0, 0, 1, 0, 0, 0)+ (2, 0, 0, 1, 3, 0)+ (0, 2, 0, 1, 1, 0)+ (0, 2, 0, 1, 0, 1) | (1, 0, 1, 4, 0, 0) (3, 2, 1, 0, 0, 0) — — | (3, 2, 1, 0, 0, 0) (0, 0, 1, 3, 1, 1) — — | (0, 1, 1, 0, 2, 2) (0, 1, 1, 0, 2, 2) — — | {4} {3} — — | (Qrt)(Cub) (Cub) ² (Con) — — |
| $\{6\}_{Quint}$ | (1, 2, 2, 2, 3, 1)[11] | (0, 0, 1, 1, 0, 0)+ (0, 1, 0, 0, 1, 0)+ (1, 0, 0, 1, 2, 0)+ (0, 1, 1, 0, 0, 1) | (3, 2, 1, 1, 0, 0) (0, 0, 2, 0, 2, 1) — — | (0, 1, 0, 2, 1, 2) (0, 1, 0, 2, 1, 2) — — | (0, 0, 2, 0, 2, 1) (2, 1, 2, 0, 1, 0) — — | {3} {2} ² — — | (Cub)(Con) ² (Con) ² — — |
| $\{7\}_{\square}$ | (1, 1, 1, 3, 1, 5)[12] | (0, 0, 0, 1, 1, 0)+ (0, 0, 1, 0, 0, 1)+ (1, 0, 0, 1, 0, 2)+ (0, 1, 0, 1, 0, 2) | (3, 3, 2, 1, 1, 0) (4, 4, 2, 0, 2, 0) — — | (0, 0, 1, 2, 0, 1) (0, 0, 0, 0, 2, 2) — — | (0, 0, 0, 0, 2, 2) (0, 0, 1, 2, 0, 1) — — | {3} {4} — — | (Cub)(Con) ² (Qrt)(Con) — — |
| $\{7\}_{Quint}$ | (1, 2, 2, 3, 4, 2)[14] | (1, 0, 0, 1, 2, 0)+ (0, 1, 0, 1, 1, 0)+ (0, 0, 1, 1, 1, 0)+ (0, 1, 1, 0, 0, 2) | (3, 2, 2, 1, 0, 0) (3, 2, 2, 1, 0, 0) — — | (0, 1, 1, 0, 2, 1) (1, 0, 0, 3, 0, 2) — — | (0, 0, 0, 2, 1, 2) (0, 1, 1, 0, 2, 1) — — | {3} {4} — — | (Cub)(Con) ² (Cub) ² — — |
| $\{7\}_{Sext}$ | (1, 2, 1, 2, 1, 2)[9] | (0, 0, 0, 0, 1, 1)+ (0, 0, 1, 1, 0, 0)+ (1, 1, 0, 0, 0, 0)+ (0, 1, 0, 1, 0, 1) | (0, 2, 1, 1, 2, 0) — — — | (0, 2, 2, 0, 1, 1) — — — | (1, 1, 0, 2, 2, 0) — — — | {2} ³ — — — | (Con) ³ — — — |

recurrence formulae valid for all d .

As an alternative to the Batyrev reflexive polyhedron method, we have proposed a new description of CY_d spaces based on the structures of the set of invariant monomials (IMs). We have shown that the IM approach, which is based on Diophantine decompositions, is a valuable alternative to the normal RWV expansion approach. We have demonstrated this by comparing the results of both approaches for the first three diagonal lines, A_r , D_r and E_r , in the arity-dimension plot for CY_3 , CY_4 cases.

We have shown that recurrence relations for conic, cubic and quartic monomials give us the formulae for the numbers of IMs in arbitrary dimensions. This was illustrated in three cases, for CY_d spaces with $\{10\}_{\Delta}$, $\{9\}_{\Delta}$ and $\{7\}_{\Delta}$ fibres. This confirms that, in the framework of the UCYA, the Calabi-Yau ‘genome’ can in principle be solved completely.

Table 19: *The CY_4 IMs with the Diophantine conditions: $\{3\} \rightarrow 1/3(P_1 + P_2 + P_3) = E_6$ and $\{4\} \rightarrow 1/4(P_1 + P_2 + 2C_2) = E_6$, $1/2(C_1 + C_2) = E_6$, $M_6 + 2P_1 = 3C_1$. Here we present some of the 84 Weierstrass sets of IMs.*

| N | $k_i(eld)$ | k^{1ex} | k^{2ex} | k^{3ex} | k^{4ex} | M_6 |
|-----|-------------------------|---------------------|---------------------|---------------------|--------------------|------------------------|
| 1 | (1, 1, 1, 1, 8, 12)[24] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (0, 0, 0, 1, 2, 3) | (6, 6, 6, 6, 0, 0) |
| 2 | (1, 1, 1, 1, 7, 11)[22] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (0, 0, 0, 1, 1, 2) | (6, 6, 6, 4, 0, 0) |
| 3 | (1, 1, 1, 1, 7, 10)[21] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (0, 0, 0, 1, 1, 1) | (6, 6, 6, 3, 0, 0) |
| 4 | (1, 1, 1, 1, 6, 10)[20] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (0, 0, 0, 1, 0, 1) | (6, 6, 6, 2, 0, 0) |
| 5 | (1, 1, 1, 1, 6, 9)[19] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (0, 0, 0, 1, 0, 0) | (6, 6, 6, 1, 0, 0) |
| 6 | (2, 1, 1, 6, 8, 6)[24] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (1, 0, 0, 3, 2, 0)+ | (0, 0, 1, 3, 2, 0) | (6, 6, 6, 0, 0, 0) |
| 6' | (2, 1, 1, 9, 8, 3)[24] | (1, 0, 0, 0, 2, 3)+ | (1, 0, 0, 3, 2, 0)+ | (0, 1, 0, 3, 2, 0)+ | (0, 0, 1, 3, 2, 0) | (6, 6, 6, 0, 0, 0) |
| 7 | (2, 1, 1, 4, 4, 12)[24] | (1, 0, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (1, 0, 0, 2, 0, 3)+ | (0, 1, 0, 2, 0, 3) | (6, 6, 6, 0, 0, 0) |
| 7' | (2, 1, 1, 3, 6, 11)[24] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (1, 0, 0, 3, 0, 2) | (6, 6, 6, 0, 0, 0) |
| 8 | (1, 1, 1, 1, 8, 12)[24] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 2, 3)+ | (0, 0, 0, 1, 2, 3) | (6, 6, 6, 6, 0, 0) |
| 9 | (1, 1, 1, 1, 6, 10)[20] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 1, 2)+ | (0, 0, 0, 1, 1, 2) | (6, 6, 4, 4, 0, 0) |
| 10 | (1, 1, 1, 1, 6, 9)[19] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 1, 2)+ | (0, 0, 0, 1, 1, 1) | (6, 6, 4, 3, 0, 0) |
| 11 | (1, 1, 1, 1, 5, 9)[18] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 1, 2)+ | (0, 0, 0, 1, 0, 1) | (6, 6, 4, 2, 0, 0) |
| 12 | (1, 1, 1, 1, 5, 8)[17] | (1, 0, 0, 0, 2, 3)+ | (0, 1, 0, 0, 2, 3)+ | (0, 0, 1, 0, 1, 2)+ | (0, 0, 0, 1, 0, 0) | (6, 6, 4, 1, 0, 0) |
| — | — | — | — | — | — | (6, ..., ..., 0, 0, 0) |
| 81 | (4, 3, 3, 3, 3, 8)[24] | (1, 3, 0, 0, 0, 2)+ | (1, 0, 3, 0, 0, 2)+ | (1, 0, 0, 3, 0, 2)+ | (1, 0, 0, 0, 3, 2) | (6, 0, 0, 0, 0, 0) |
| 82 | (4, 4, 4, 6, 3, 3)[24] | (2, 0, 1, 3, 0, 0)+ | (0, 2, 1, 3, 0, 0)+ | (0, 2, 1, 0, 3, 0)+ | (2, 0, 1, 0, 0, 3) | (0, 0, 6, 0, 0, 0) |
| 82' | (6, 2, 4, 6, 3, 3)[24] | (2, 0, 1, 3, 0, 0)+ | (0, 2, 1, 3, 0, 0)+ | (2, 0, 1, 0, 3, 0)+ | (2, 0, 1, 0, 0, 3) | (0, 0, 6, 0, 0, 0) |
| 83 | (4, 2, 2, 4, 9, 3)[24] | (2, 0, 0, 1, 3, 0)+ | (0, 0, 2, 1, 3, 0)+ | (2, 0, 0, 1, 3, 0)+ | (0, 2, 0, 1, 0, 3) | (0, 0, 0, 6, 0, 0) |
| 83' | (4, 2, 2, 4, 9, 3)[24] | (2, 0, 0, 1, 3, 0)+ | (0, 2, 0, 1, 3, 0)+ | (0, 0, 2, 1, 3, 0)+ | (2, 0, 0, 1, 0, 3) | (0, 0, 0, 6, 0, 0) |
| 84 | (1, 1, 1, 1, 2, 6)[12] | (2, 0, 0, 0, 1, 3)+ | (0, 2, 0, 0, 1, 3)+ | (0, 0, 2, 0, 1, 3)+ | (0, 0, 0, 2, 1, 3) | (0, 0, 0, 0, 6, 0) |

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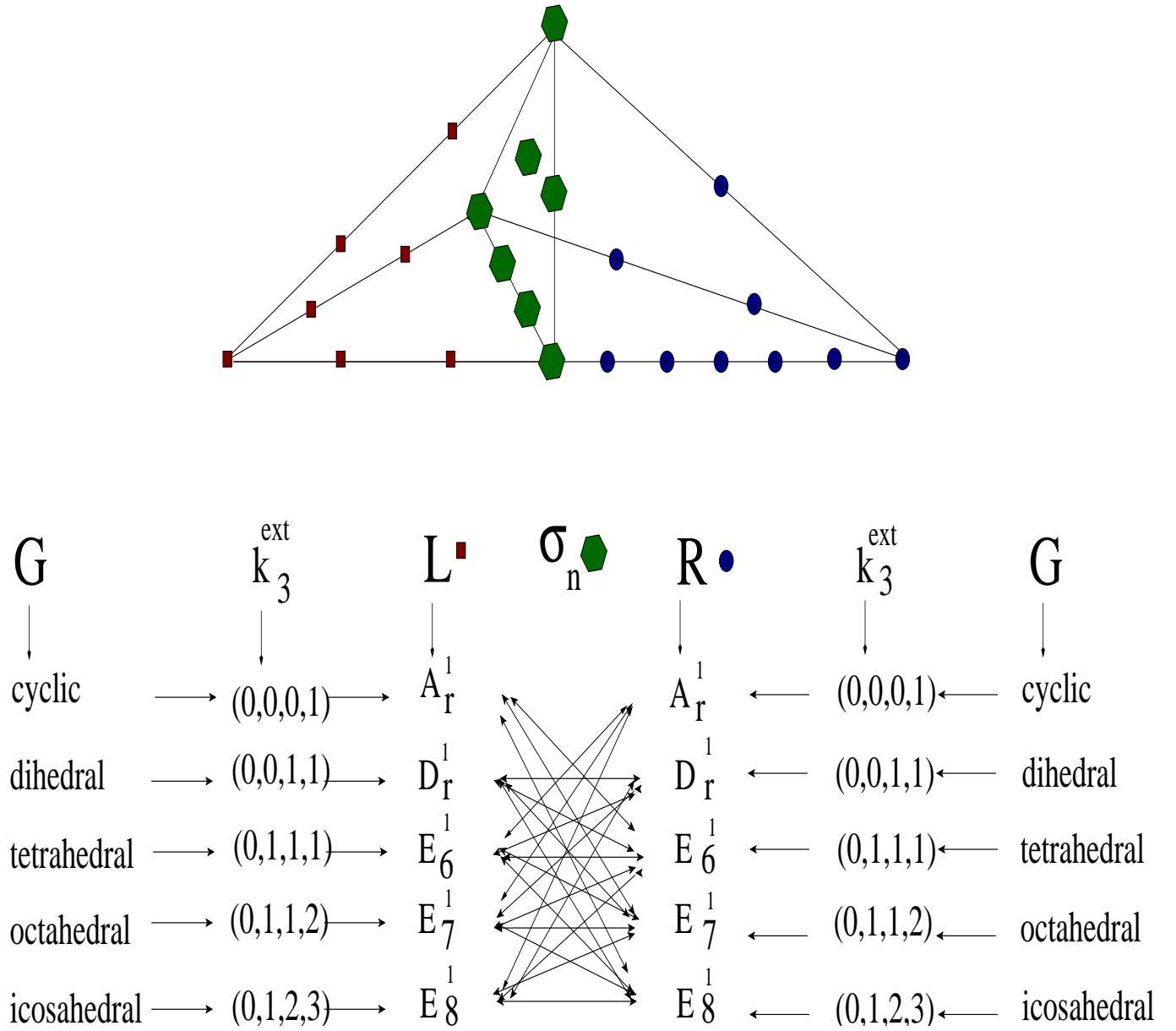


Figure 1: The K3 polyhedron determined by the reflexive weight vector $\vec{k}_4 = (1, 1, 3, 4)$ [9], which illustrates the appearance of Coxeter-Dynkin diagrams. The intersection σ is determined by 7 point monomials that correspond to the elliptic fibre $\{7\}_\Delta$, and divides the polyhedron into 7(+ 3) points on the left and 9(+7) on the right. These reproduce the Coxeter-Dynkin diagrams for affine E_6 and E_8 , respectively. Underneath, we also show schematically the general nature of the highest-weight vectors obtained by arity-2 construction in the UCYA, displaying the one-to-one link between the 5-dimensional weight vectors and the ADE series of Cartan-Lie algebra in K3 hypersurfaces. The rôles of the discrete symmetry groups were discussed in [34]

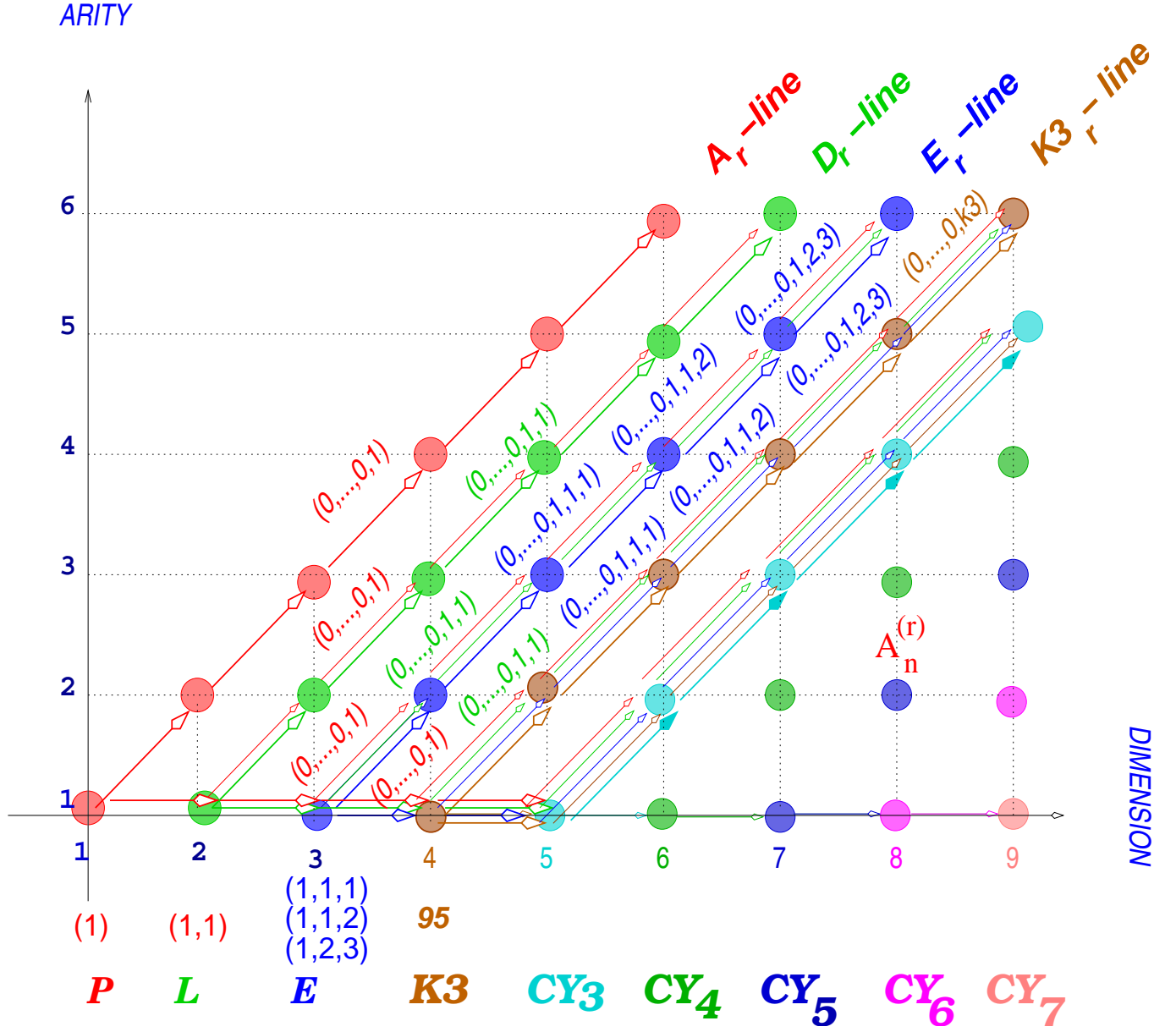


Figure 2: The arity-dimension plane, illustrating the normal expansion of RWVs by adding zero components to lower-dimensional vectors. For each dimension n and arity r , it is possible to reconstruct a set of extended vectors $\{\mathbb{E}_n^{(r)}\}$. Along the A_r line, one adds zeroes to the trivial vector (1) , along the D_r line also to the vector $(1,1)$, etc.. Along the $K3_r$ line, one may also add zeroes to any of the 95 $K3$ vectors [34, 35].

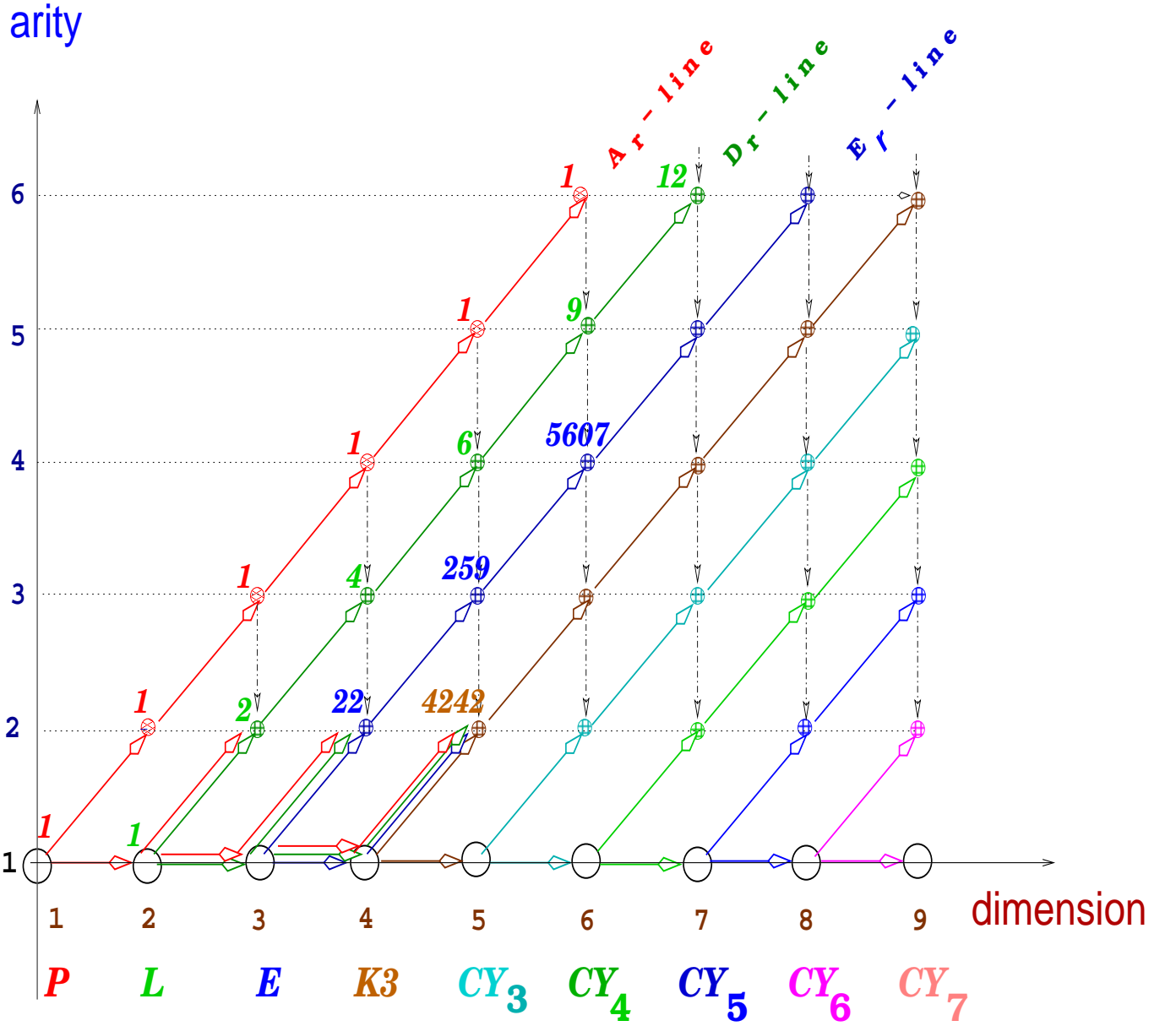


Figure 3: The arity-dimension plot, showing the numbers of eldest vectors/chains obtained by normal extensions of RWVs, including previous results for CY_3 and lower-dimensional spaces, and new results for CY_4 and CY_5 spaces.

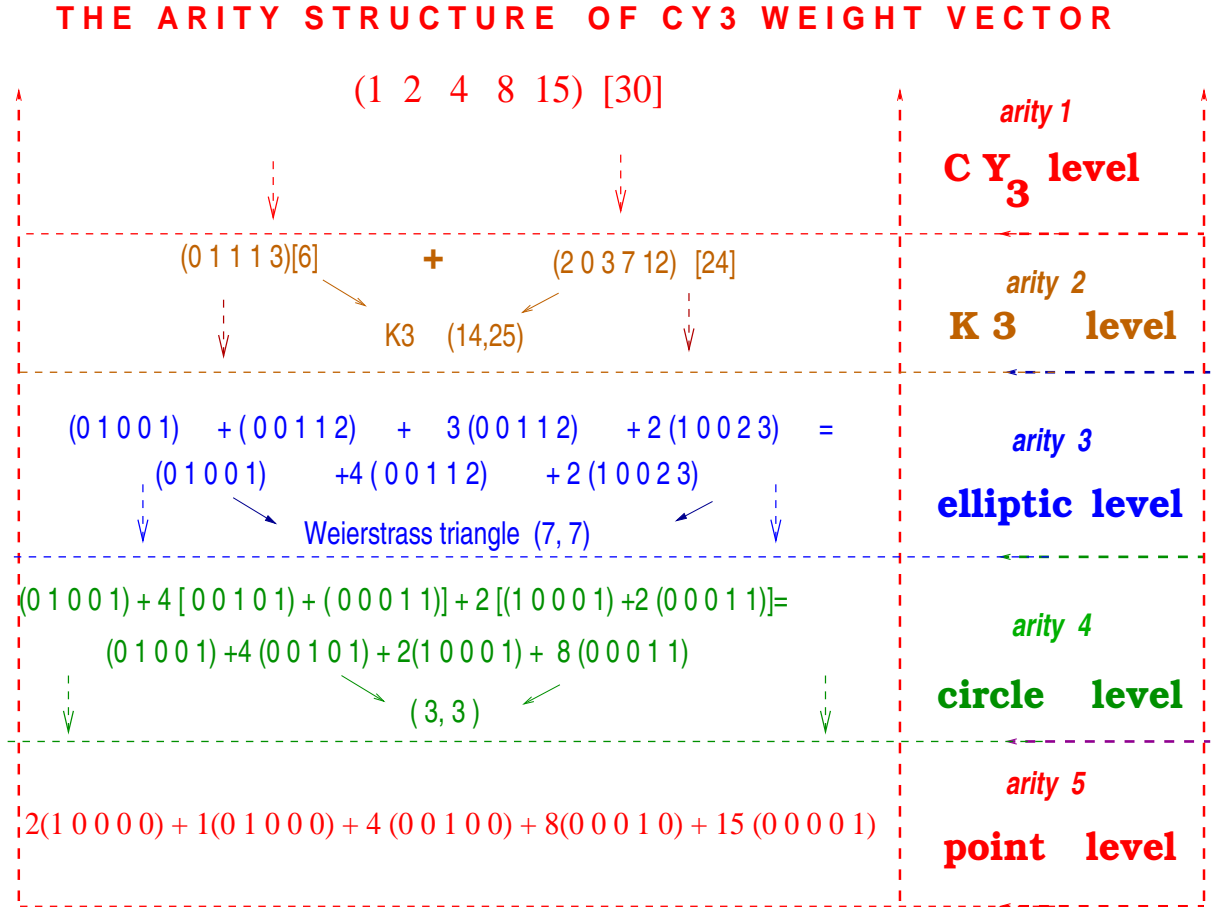


Figure 4: An example of the arity structure of a sample CY_3 weight vector, obtainable by normal expansion of higher-arity RWVs, showing the relation to specific $K3$ and Weierstrass spaces.

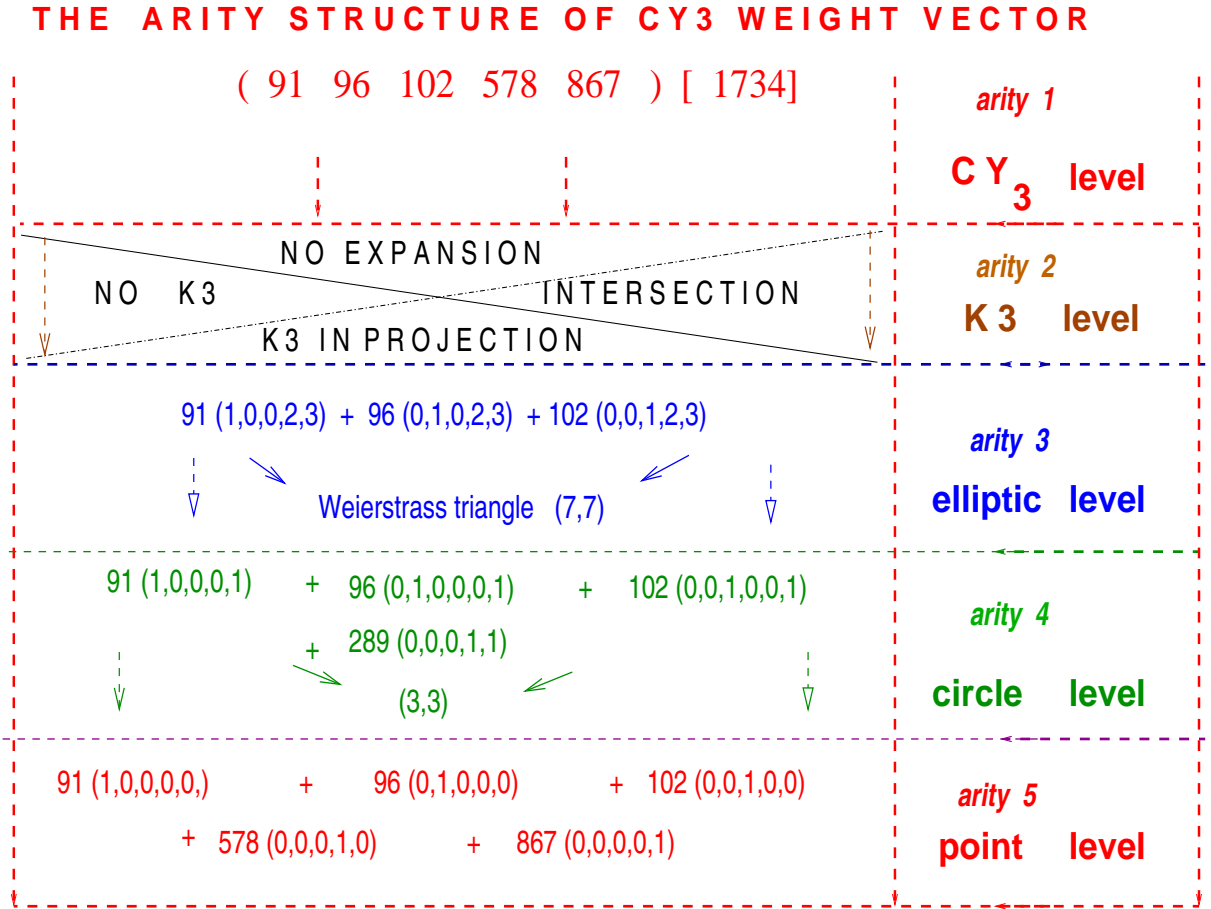


Figure 5: Another example of the arity structure of a sample CY₃ weight vector, obtainable by normal expansion of higher-arity RWVs, which has counterparts at arities 3, 4 and 5, but not arity 2.

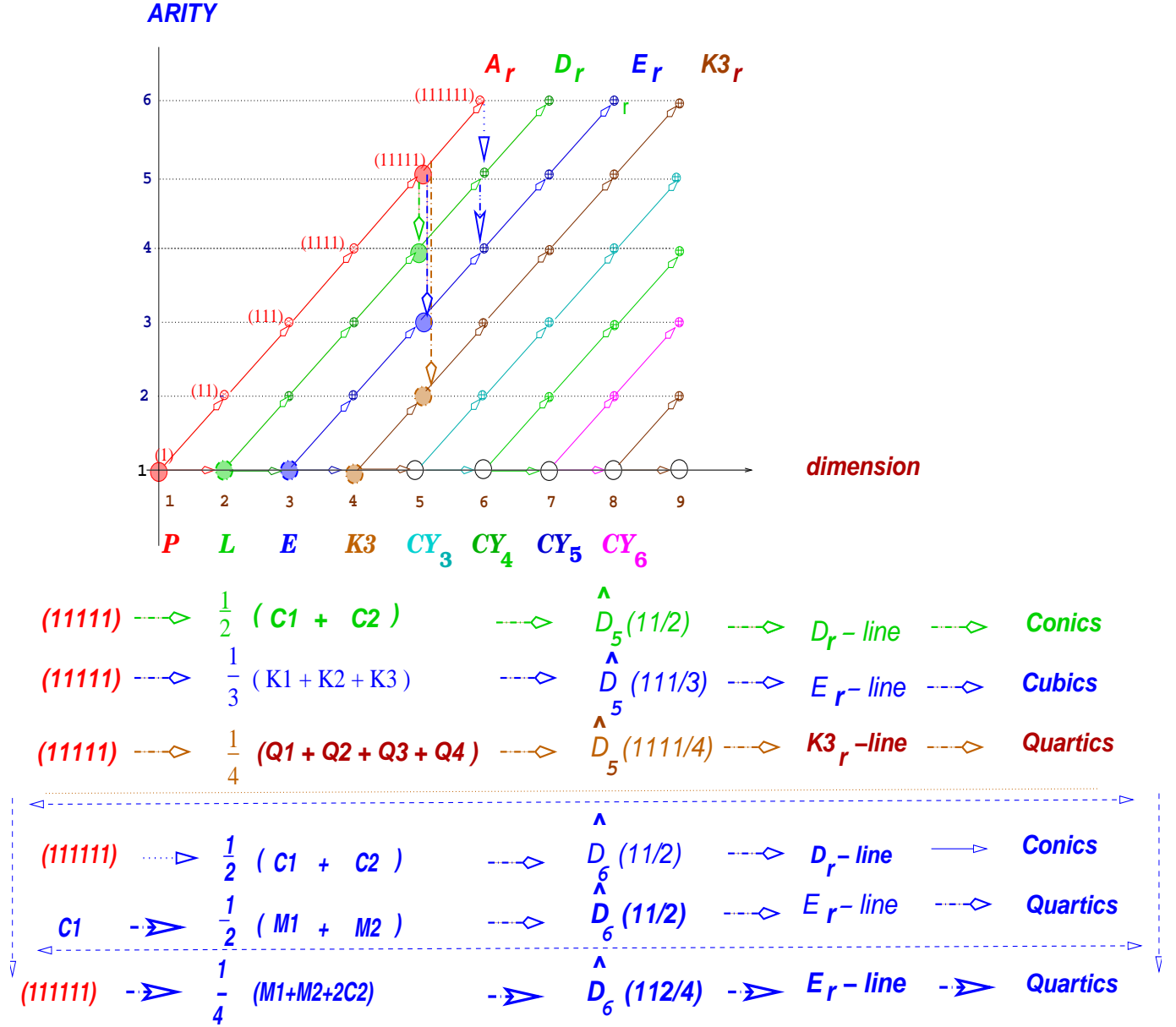


Figure 6: Examples of the Diophantine decompositions of unit monomials, illustrating how conics, cubics and quartics may be used to construct spaces with lower arities.

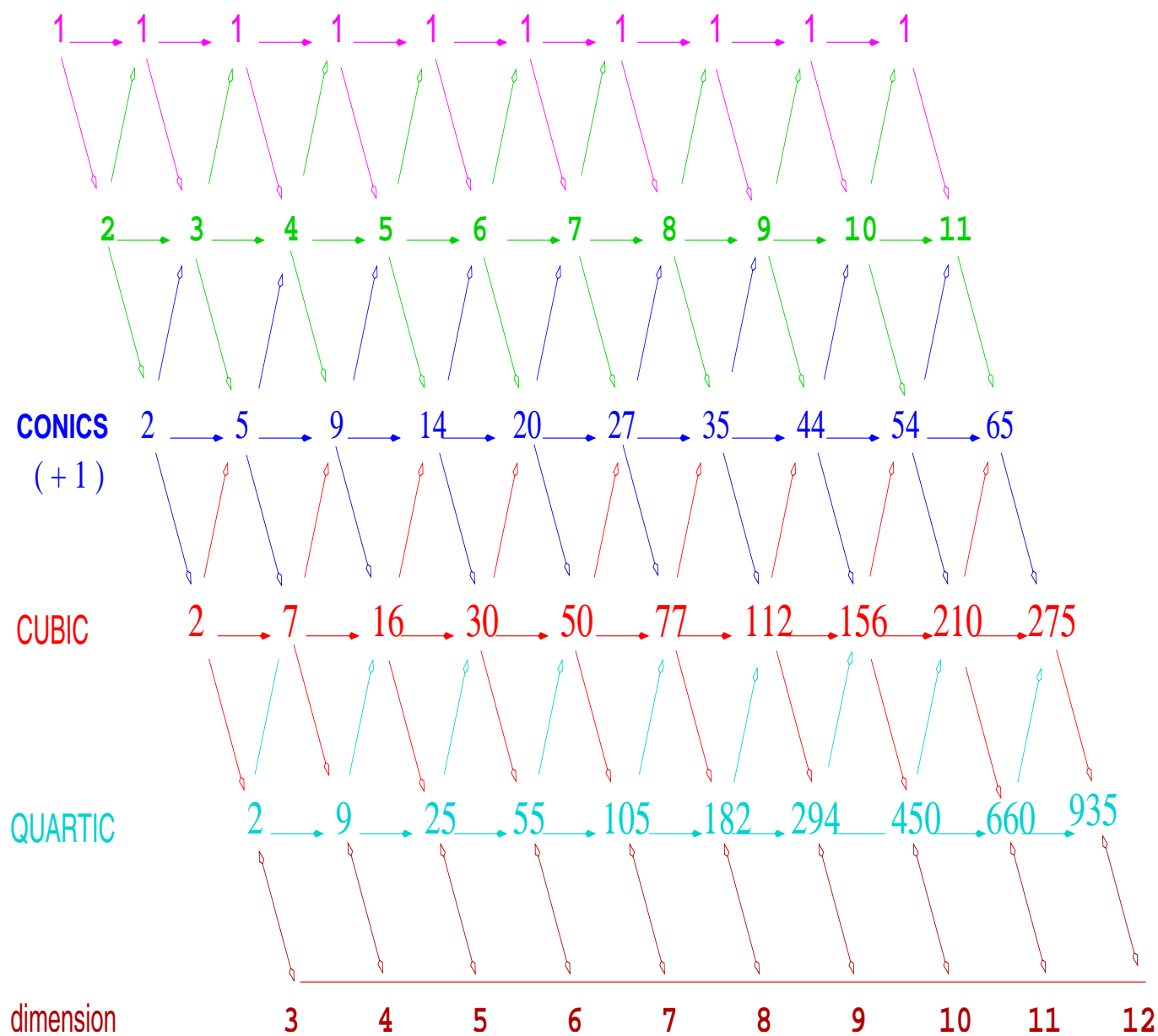


Figure 7: *Lattice illustrating recurrence relations for the numbers of conic, cubic and quartic monomials.*

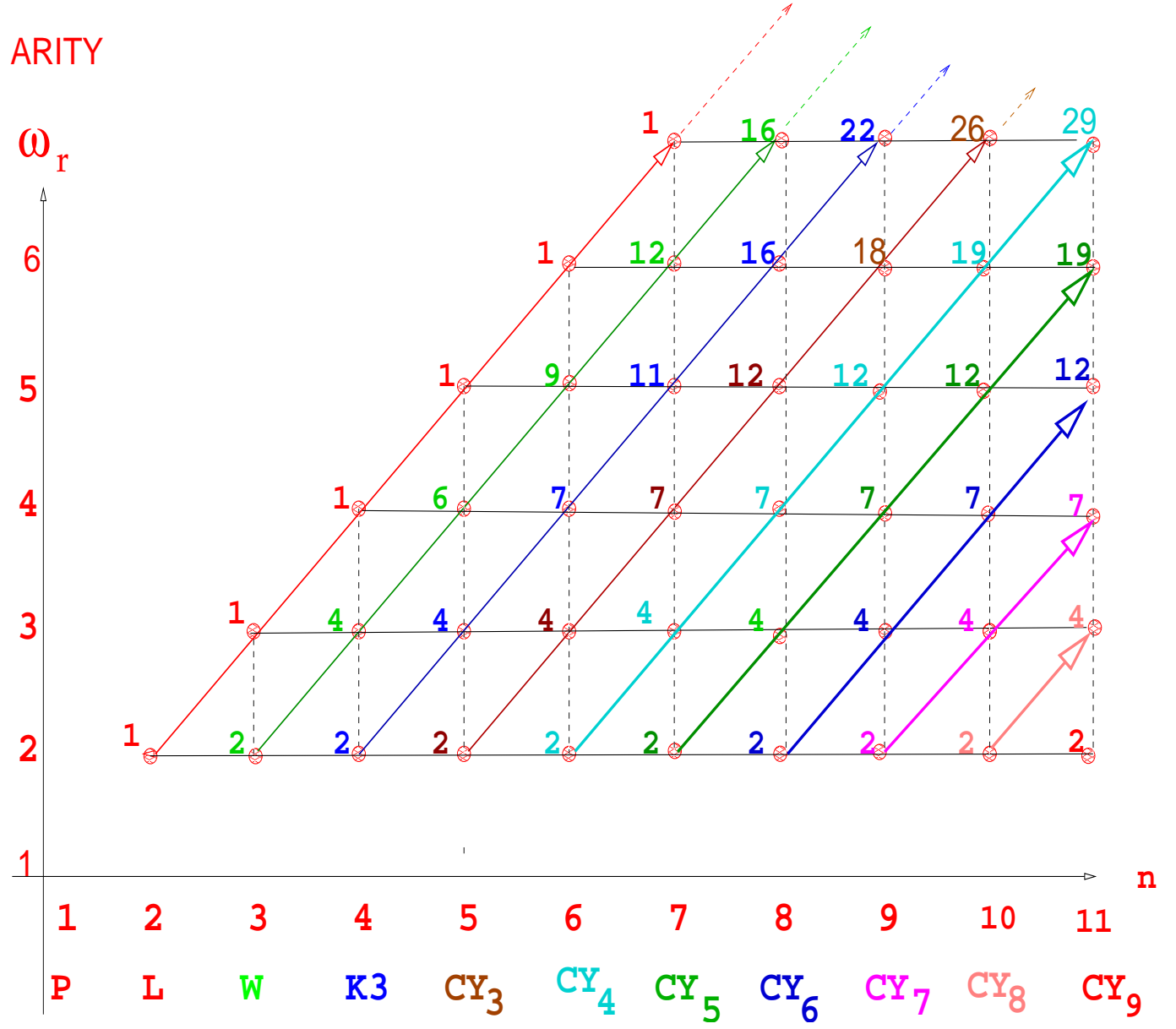


Figure 8: The numbers of recurrences of Calabi-Yau hypersurfaces with a $(1, \dots, 1)_n$ fibre are calculable along all lines $n = r + p$ in the arity-dimension plot.